Problem 2. Thinking outside the conducting sphere. *Solution.*

a) [3.5 points] Because the conducting sphere is grounded, we can treat the two-point particles as non-interacting. To compute the forces, we thus wish to compute the electric field inside and outside the conductor, and, for this, we will use the method of images. We wish to find the image charges for the inside and outside charges such that the electric potential vanishes on the surface of the sphere. This is achieved by placing the image charges between the center of the sphere and the inside/outside charges at a location $\vec{p}$ with a charge $\tilde{q}$ for the image of the inside charge and at a location $\vec{p}'$ with charge $\tilde{q}'$ for the outside charges. The potential on the inside of the sphere at a location $\vec{x}$ ($|\vec{x}| < R$)is given by

$$V_{\text{inside}}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\vec{x}-\vec{r}|} + \frac{\tilde{q}}{|\vec{x}-\vec{p}|}\right]. \tag{2.1}$$

Imposing that this potential vanishes on the surface of the sphere, we find *(1 point)*

$$\tilde{q} = -q\frac{R}{r}, \qquad \vec{p} = \frac{R^2}{|\vec{r}|}\hat{r},$$

which thus lies outside the sphere. Similarly, on the outside, at a location $\vec{x}$ ($|\vec{x}| > R$), we have

$$V_{\text{outside}}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q'}{|\vec{x}-\vec{r}'|} + \frac{\tilde{q}'}{|\vec{x}-\vec{p}'|}\right]. \tag{2.2}$$

Once again imposing that this vanishes on the surface of the sphere, we find *(1 point)*

$$\tilde{q}' = -q'\frac{R}{r'}, \qquad \vec{p}' = \frac{R^2}{|\vec{r}'|}\hat{r}'.$$

which thus lies inside the sphere. The force on the particle inside is simply given by the attraction to the image charge outside, while the force on the particle outside is given by the attraction to the image charge inside. The former is given by *(0.75 points)*

$$\vec{F}_{\text{inside}} = \frac{1}{4\pi\epsilon_0}\frac{q^2 R}{|\vec{r}|^3\left(\frac{R^2}{|\vec{r}|^2}-1\right)^2}\hat{r},$$

while the latter is *(0.75 points)*

$$\vec{F}_{\text{outside}} = -\frac{1}{4\pi\epsilon_0}\frac{(q')^2 R}{|\vec{r}'|^3\left(1-\frac{R^2}{|\vec{r}'|^2}\right)^2}\hat{r}'.$$

b) [3.5 points] One approach to generalize the solution above when we replace the point particles with spheres is to find the image charges for each charge element dq that is part of the interior of the two homogeneously charged spheres. This is, however, unnecessarily complicated. Instead, we can notice that because the contribution to the electric potential from a homogeneously charged sphere is the same as that of a charged particle placed at the center of the sphere (with a charge $q = \frac{4}{3}\pi a^3\rho$ for the inside and a charge $q' = \frac{4}{3}\pi(a')^3\rho'$ for the outside) the same image charges studied in part a) make the potential vanish on the surface of the conductor! Thus, outside the volumes of the homogeneously charged spheres, the electric potentials (2.1) and (2.2) found in point a) are unchanged (*2.0 points*). The forces acting on the inside and outside spheres are, thus, again (*1.5 points*)

$$\vec{F}_{\text{inside}} = \frac{1}{4\pi\epsilon_0}\frac{q^2 R}{|\vec{r}|^3\left(\frac{R^2}{|\vec{r}|^2}-1\right)^2}\hat{r}, \qquad \vec{F}_{\text{outside}} = -\frac{1}{4\pi\epsilon_0}\frac{(q')^2 R}{|\vec{r}'|^3\left(1-\frac{R^2}{|\vec{r}'|^2}\right)^2}\hat{r}'.,$$

where

$$q = \frac{4}{3}\pi a^3\rho, \qquad q' = \frac{4}{3}\pi(a')^3\rho'. \tag{2.3}$$

c) [3 points] Because the particles (or spheres) do not interact, we can use conservation of energy separately for the charged particles (or spheres) inside or outside the conductor. Since the force is always in the direction $\widehat{r}_0$ for the particle or sphere inside (and in the direction $\widehat{r}'_0$ for the particle or sphere outside) the only component of the velocity will be along the $\widehat{r}_0$ direction (or the $\widehat{r}'_0$ direction). For the particle or sphere inside, conservation of energy tells us that (*1.0 points*)

$$\frac{m\dot{r}^2(t)}{2} = \int_{\vec{r}_0}^{\vec{r}} \vec{F}_{\text{inside}}(\vec{x})d\vec{x} = \frac{1}{8\pi\epsilon_0}\frac{q^2R}{R^2-r^2} - \frac{1}{8\pi\epsilon_0}\frac{q^2R}{R^2-r_0^2} \qquad \Rightarrow \qquad \dot{r} = \left[\frac{q^2R}{4\pi\epsilon_0\, m}\left(\frac{1}{R^2-r^2} - \frac{1}{R^2-r_0^2}\right)\right]^{\frac{1}{2}}. \tag{2.4}$$

When the inside sphere hits the surface of the conductor, we have (*0.5 point*)

$$\dot{r}_{\text{sph. inside}} = \left[\frac{q^2R}{4\pi\epsilon_0\, m}\left(\frac{1}{R^2-(R-a)^2} - \frac{1}{R^2-r_0^2}\right)\right]^{\frac{1}{2}}. \tag{2.5}$$

Note that if the point particle were to hit the conductor ($a \to 0$) its velocity would be infinite and we could not neglect the effect of magnetic induction.

For the particle or sphere outside, we similarly have (*1.0 point*)

$$\frac{m(\dot{r}'(t))^2}{2} = \int_{\vec{r}'_0}^{\vec{r}'} \vec{F}_{\text{outside}}(\vec{x})d\vec{x} = \frac{1}{8\pi\epsilon_0}\frac{q^2R}{(r')^2-R^2} - \frac{1}{8\pi\epsilon_0}\frac{q^2R}{(r'_0)^2-R^2} \qquad \Rightarrow$$

$$\Rightarrow \quad \dot{r}' = -\left[\frac{q^2R}{4\pi\epsilon_0\, m}\left(\frac{1}{(r')^2-R^2} - \frac{1}{(r'_0)^2-R^2}\right)\right]^{\frac{1}{2}}. \tag{2.6}$$

When the sphere touches the conductor (*0.5 points*), we have

$$\dot{r}_{\text{sph. outside}} = -\left[\frac{q^2R}{4\pi\epsilon_0\, m}\left(\frac{1}{(R+a)^2-R^2} - \frac{1}{(r'_0)^2-R^2}\right)\right]^{\frac{1}{2}}. \tag{2.7}$$

The velocity would once again be infinite when the sphere would be replaced by a point particle.