Theoretical problem 2: The adiabatic piston

A vertical cylinder with adiabatic walls contains argon, considered as an ideal gas (the molar mass of argon is $\mu = 40\ \mathrm{g/mol}$). The gas is contained in the cylinder with the help of an adiabatic piston, the mass of which is $m = 0.64\ \mathrm{kg}$. The inner cross-section area of the cylinder is $A = 80\ \mathrm{cm}^2$. The piston can move without friction inside the cylinder and does not let the gas escape. The air surrounding the cylinder – piston system is evacuated. Initially, the gas temperature is $T_i = 293\ \mathrm{K}$ and the height at which the piston is in static equilibrium, measured as the height of the gas column in the cylinder, is $y_i = 1.0\ \mathrm{m}$. The gravitational acceleration is $g = 9.81\ \mathrm{m/s^2}$ and the universal constant of ideal gases is $R = 8.315\ \mathrm{J\ mol^{-1}\ K^{-1}}$.

A. Gas parameters in the initial state [0.8 points]

A	Derive the mathematical expressions and calculate the numerical values for the gas pressure in the initial equilibrium state and for the mass of argon in the cylinder.	**0.8 points**

Solution: The initial pressure of the gas is

$$\boxed{p_i = \frac{mg}{A}}, \qquad \textbf{[0.2 p]}$$

having the numerical value

$$p_i = 785\ \mathrm{Pa} \cong 7.9 \cdot 10^2\ \mathrm{Pa}. \qquad \textbf{[0.1 p]}$$

The amount of substance, according to Clapeyron-Mendeleev eq., has the expression

$$m_{Ar} = \frac{p_i V_i \mu}{R T_i}. \qquad \textbf{[0.2 p]}$$

Since

$$V_i = A y_i, \qquad \textbf{[0.1 p]}$$

then

$$\boxed{m_{Ar} = \frac{m g y_i \mu}{R T_i}}, \qquad \textbf{[0.1 p]}$$

having the numerical value

$$m_{Ar} = 1{,}03 \cdot 10^{-4}\ \mathrm{kg} \cong 1{,}0 \cdot 10^{-4}\ \mathrm{kg}. \qquad \textbf{[0.1 p]}$$

B. Very small perturbations [2.3 points]

The purpose of this problem is to find out what happens with the gas and the piston after perturbing the piston. To do this, the piston is suddenly pushed, such that its initial speed u_i is so much smaller than the thermal speed v_{T_i} of the gas in the initial state that their ratio can be neglected.

B1	Derive the mathematical dependence of the gas temperature T of y.	**0.4 points**

Solution:

Let's suppose, for fixing the ideas, that the piston moves upwards. The piston moves so slowly that the gas expansion is *quasistatic*, so each state the gas passes through is an equilibrium state. For the quasistatic adiabatic expansion, the Poisson equation is valid:

$$T_i V_i^{\gamma-1} = T V^{\gamma-1}, \qquad \textbf{[0.2 p]}$$

or

$$T_i y_i^{\gamma-1} = T y^{\gamma-1}. \qquad \textbf{[0.1 p]}$$

Hence

$$\boxed{T = T_i \left(\frac{y_i}{y}\right)^{2/3}}, \qquad \textbf{[0.1 p]}$$

because the argon is a monoatomic gas ($\gamma = 5/3$).

Note: *To prove that the gas obeys the Poisson law under the adopted assumption, let's write the eq. of motion for the piston and the 1st law of thermodynamics:*

$$m\frac{du}{dt} = pA - mg = \frac{pV}{y} - mg = \frac{nRT}{y} - mg \qquad (*)$$

and

$$\frac{m}{2}\left(u^2 - u_i^2\right) + nC_V(T - T_i) = -mg(y - y_i),$$

which, differentiated, gives

$$mudu + \frac{1}{\gamma - 1} nRdT = -mgdy,$$

or

$$m\frac{du}{dt} = -\frac{1}{\gamma-1} nR\frac{dT}{dy} - mg. \qquad (**)$$

From (*) and (**), it follows that

$$\frac{nRT}{y} + \frac{1}{\gamma - 1} nR\frac{dT}{dy} = 0,$$

or

$$(\gamma - 1)\frac{dy}{y} + \frac{dT}{T} = 0,$$

which, by integration, gives

$$Ty^{\gamma-1} = const.$$

B2	Derive the equation describing the position of the piston, with respect to the bottom of the cylinder, as a function of time, $y(t)$.	**1.6 points**

Solution:

Considering that the piston move upwards, from the eq. of motion of the piston

$$ma = pA - mg, \qquad \textbf{[0.2 p]}$$

where

$$mg = p_i A \qquad \textbf{[0.1 p]}$$

and from Poisson eq.

$$p = p_i\left(\frac{y_i}{y}\right)^{\gamma} = p_i\left(\frac{y_i}{y_i+\Delta y}\right)^{\gamma} = p_i\left(1+\frac{\Delta y}{y_i}\right)^{-\gamma} \cong p_i\left(1-\gamma\frac{\Delta y}{y_i}\right), \qquad \textbf{[0.4 p]}$$

because the perturbation is very small.

So,

$$a = -\frac{\gamma g}{y_i}\Delta y, \qquad \textbf{[0.1 p]}$$

meaning that the movement of the piston is a harmonic oscillatory one, described by the eq.

$$y = y_i + B cos(\omega_0 t + \varphi_0), \qquad \textbf{[0.2 p]}$$

where

$$\omega_0 = \sqrt{\frac{\gamma g}{y_i}}, \qquad \textbf{[0.1 p]}$$

which means that the oscillations frequency is $f_0 = \frac{1}{2\pi}\sqrt{\frac{\gamma g}{y_i}} = 0.64$ Hz.

For finding B and φ_0, the initial conditions must be used:

$$y(0) = y_i \text{ and } u(0) = u_i \qquad \textbf{[2 x 0.1 p]}$$

The results are:

$$\varphi_0 = \frac{\pi}{2} \text{ and } A = -\frac{u_i}{\omega_0} = -u_i\sqrt{\frac{y_i}{\gamma g}}. \qquad \textbf{[2 x 0.1 p]}$$

So,

$$\boxed{y = y_i\left[1+\frac{u_i}{\sqrt{\gamma g y_i}} sin\left(\sqrt{\frac{\gamma g}{y_i}}t\right)\right]}. \qquad \textbf{[0.1 p]}$$

Note: The ratio $\frac{u_i}{\sqrt{\gamma g y_i}}$ cannot be neglected because

$$\frac{1}{\sqrt{\gamma g y_i}} \cong 0.25\,\frac{\text{s}}{\text{m}}.$$

B3	Calculate the entropy variation of the gas in this process.	0.3 points

Solution:

For any quasistatic adiabatic process

$$\boxed{\Delta S = 0}. \qquad \textbf{[0.3 p]}$$

C. Small perturbations [6.9 points]

The speed u of the piston is at least one order of magnitude higher than in the previous part of the problem, but still much smaller than the thermal speed of the gas. Due to the smallness of the piston's speed, there is no turbulence inside the gas. Moreover, it will be considered that the pressure is not the same everywhere in the gas volume. When the piston moves upwards, for example, a rarefaction thin layer will appear immediately under the piston. To make a simple model, we will divide the gas volume in two hydrodynamic regions:

- the rarefaction thin layer, situated immediately under the piston, having the following properties:
  - its volume (δV) is negligible when compared to the entire gas volume;
  - the pressure is uniform in the layer, but slightly smaller than in the rest of the gas;
  - the temperature is the same as in the rest of the gas;
- the rest of the gas in which the pressure and the temperature are those of equilibrium.

During the piston movement, its interaction with the gas, mediated by the hydrodynamic layer beneath it, has a macroscopic effect: the appearance of a drag force acting on the piston. In this way the mechanical energy of the piston is degraded into thermal energy and the piston will eventually stop moving.

C1	Derive the expression for the final position of the piston (y_f), measured with respect to the bottom of the cylinder, as well as for the final temperature of the gas (T_f). Calculate their numerical values considering $u_i = 5.0$ m/s.	**1.6 points**

Solution:

The 1st law of Thermodynamics has the form

$$\frac{m}{2}\left(u_f^2 - u_i^2\right) + \Delta U = L_{ext}, \qquad \textbf{[0.2 p]}$$

where

$$L_{ext} = -mg\left(y_f - y_i\right) \qquad \textbf{[0.1 p]}$$

and

$$\Delta U = nC_V\left(T_f - T_i\right) = \frac{3}{2}nR\left(T_f - T_i\right) = \frac{3}{2}\left(p_f V_f - p_i V_i\right). \qquad \textbf{[0.3 p]}$$

In the final state

$$u_f = 0 \qquad \textbf{[0.1 p]}$$

and

$$p_f = p_i = \frac{mg}{A}, \qquad \textbf{[0.1 p]}$$

so

$$\Delta U = \frac{3}{2}\frac{mg}{A}\left(Ay_f - Ay_i\right) = \frac{3}{2}mg\left(y_f - y_i\right). \qquad \textbf{[0.2 p]}$$

As a result,

$$\boxed{y_f = y_i\left[1 + \frac{u_i^2}{5gy_i}\right]}. \qquad \textbf{[0.1 p]}$$

Numerically,

$$y_f = 1.51 \text{ m} \cong 1.5 \text{ m}. \qquad \textbf{[0.1 p]}$$

Because

$$\frac{nRT_f}{V_f} = \frac{nRT_i}{V_i} \rightarrow T_f = T_i \frac{y_f}{y_i},$$ **[0.2 p]**

then

$$\boxed{T_f = T_i \left[1 + \frac{u_i^2}{5gy_i}\right]}.$$ **[0.1 p]**

Numerically,

$$T_f = 442 \text{ K} \cong 440 \text{ K}.$$ **[0.1 p]**

C2	Evaluate the variation of entropy in this process ΔS and calculate its numerical value.	**0.8 points**

Solution:

$$\Delta S = nC_V ln\frac{T_f}{T_i} + nRln\frac{V_f}{V_i} = \frac{3}{2}nRln\frac{y_f}{y_i} + nRln\frac{y_f}{y_i} = \frac{5}{2}nRln\frac{y_f}{y_i}.$$ **[0.4 p]**

Taking into account that

$$nR = \frac{p_i V_i}{T_i} = \frac{\frac{mg}{A}Ay_i}{T_i} = \frac{mgy_i}{T_i}$$ **[0.2 p]**

and that

$$\frac{y_f}{y_i} = 1 + \frac{u_i^2}{5gy_i},$$

then

$$\boxed{\Delta S = \frac{5}{2}\frac{mgy_i}{T_i} ln\left[1 + \frac{u_i^2}{5gy_i}\right]}.$$ **[0.1 p]**

Numerically,

$$\Delta S = 2.2 \cdot 10^{-2} \text{ J/K}.$$ **[0.1 p]**

The strong inequality $u \ll v_T$ means that the integer powers greater than one of the ratio u/v_T can be neglected.

C3	Evaluate the actual pressure p^* in the gas layer beneath the piston during its movement upwards.	**1.6 points**

Solution:

The requested pressure is

$$p^* = \frac{\langle F \rangle}{A},$$ **[0.1 p]**

where the average force with which the atoms act on the piston is

$$\langle F \rangle = \frac{\delta N \cdot \Delta p}{\delta t}.$$ **[0.1 p]**

Here, δt is a very short time interval, such that the speed u of the piston keeps its value, Δp is the momentum variation of one atom during one elastic collision with the piston, and δN is the number of collisions with the piston in the above time interval.

To evaluate δN, we should consider that half of the atoms move in the same direction with the piston (Joule's hypothesis) and that the relative speed of an atom with respect to the piston is $v-u$, where it was supposed that u and v have the same direction. So,

$$\delta N = \frac{1}{2}\frac{N}{V}A(v-u)\delta t. \qquad \textbf{[0.3 p]}$$

Since the atom's speed after the collision with the piston is

$$v' = v - 2u,$$

With respect to the cylinder, then, its momentum variation is

$$\Delta p = 2m_0(v-u), \qquad \textbf{[0.1 p]}$$

where m_0 is the mass of one atom.

Under these conditions,

$$\langle F\rangle = \langle\frac{1}{2}\frac{N}{V}A(v-u)2m_0(v-u)\rangle = m_0\frac{N}{V}A\langle(v-u)^2\rangle = m_0\frac{N}{V}A\left\langle v^2\left(1-\frac{u}{v}\right)^2\right\rangle \cong$$

$$\cong m_0\frac{N}{V}A\left\langle v^2\left(1-2\frac{u}{v}\right)\right\rangle = m_0\frac{N}{V}A(\langle v^2\rangle - 2u\langle v\rangle), \qquad \textbf{[0.2 p]}$$

Since $\langle v^2\rangle = \frac{RT}{\mu}$ and $\langle v\rangle = \sqrt{\frac{2RT}{\pi\mu}}$, then

$$p^* = m_0\frac{N}{V}\left(\frac{RT}{\mu} - 2u\sqrt{\frac{2RT}{\pi\mu}}\right) = m_0\frac{N}{V}\frac{RT}{\mu}\left(1-2u\sqrt{\frac{2\mu}{\pi RT}}\right) = \frac{m_{Ar}}{\mu}\frac{RT}{V}\left(1-2u\sqrt{\frac{2\mu}{\pi RT}}\right). \qquad \textbf{[0.2 p]}$$

Because

$$\frac{m_{Ar}}{\mu}RT = p^*\delta V + p(V-\delta V) = pV - (p-p^*)\delta V = pV - \delta p\delta V \cong pV, \qquad \textbf{[0.5 p]}$$

then

$$\boxed{p^* = p\left(1-2u\sqrt{\frac{2\mu}{\pi RT}}\right) = p\left(1-2\sqrt{\frac{6}{\pi}}\frac{u}{v_T}\right).} \qquad \textbf{[0.1 p]}$$

An exact analysis of the gas and the piston dynamics imply the simultaneous solution of two coupled nonlinear differential equations, a task possible only with the help of a computer. However, the problem can be solved analytically in the asymptotic approximation, that is by analyzing the final stage of the piston movement, when $y = y_f + z$, $z \ll y_f$. The physical significance of y and y_f are those from above. Also, in the final stage of the dynamics, $\frac{mu^2}{m_{Ar}v_{Tf}^2}$ can be neglected because $u \ll u_i$, even if initially, $u_i \lesssim \sqrt{\frac{m_{Ar}}{m}}v_{T_i}$.

C4	Prove that, in the final stage, the movement of the piston is an oscillatory one with damping. Derive the mathematical expression and calculate the numerical value of these oscillations' pseudoperiod.	**2.4 points**

Solution:

The equation of motion for the piston is

$$ma = p^*A - mg = \frac{p^*V}{y} - mg = \frac{m_{Ar}RT}{\mu y}\left(1-2u\sqrt{\frac{2\mu}{\pi RT}}\right) - mg, \qquad \textbf{[0.2 p]}$$

or

$$a = \frac{m_{Ar}RT}{\mu m y}\left(1-2u\sqrt{\frac{2\mu}{\pi RT}}\right) - g. \qquad \textbf{[0.1 p]}$$

The motion of the piston is damped because the gas pressure in the adjacent layer and, consequently, the force acting on the piston, has a drag component proportional with the piston's speed (and also with the gas temperature, making the force a little more complicated)!
The gas temperature close to the final state of the system can be found from the energy balance:

$$\frac{m}{2}\Big(\underbrace{u_f^2}_{=0} - u^2\Big) + nC_V(T_f - T) = -mg(y_f - y), \qquad \textbf{[0.1 p]}$$

or, because $C_V = 3R/2$,

$$T = T_f\left(1 - \frac{mu^2}{3nRT_f} - \frac{2mgz}{3nRT_f}\right) = T_f\left(1 - \frac{mu^2}{m_{Ar}v_{Tf}^2} - \frac{2p_f Az}{3p_f V_f}\right). \qquad \textbf{[0.2 p]}$$

Neglecting the ratio $\frac{mu^2}{m_{Ar}v_{Tf}^2}$ as was stated in the text,

$$T \cong T_f\left(1 - \frac{2}{3}\frac{z}{y_f}\right). \qquad \textbf{[0.1 p]}$$

Because

$$u\sqrt{\frac{2\mu}{\pi RT}} = u\sqrt{\frac{2\mu}{\pi RT_f}}\left(1 - \frac{2}{3}\frac{z}{y_f}\right)^{-\frac{1}{2}} \cong \sqrt{\frac{6}{\pi}}\frac{u}{v_{Tf}}\left(1 + \frac{1}{3}\frac{z}{y_f}\right) = \sqrt{\frac{6}{\pi}}\left(\frac{u}{v_{Tf}} + \frac{1}{3}\frac{u}{v_{Tf}}\frac{z}{y_f}\right) \cong \sqrt{\frac{6}{\pi}}\frac{u}{v_{Tf}},$$

or

$$u\sqrt{\frac{2\mu}{\pi RT}} \cong \sqrt{\frac{6}{\pi}}\frac{u}{v_{Tf}}, \qquad \textbf{[0.4 p]}$$

then

$$a \cong \frac{m_{Ar}RT_f\left(1 - \frac{2}{3}\frac{z}{y_f}\right)}{\mu m y_f\left(1 + \frac{z}{y_f}\right)}\left(1 - 2\sqrt{\frac{6}{\pi}}\frac{u}{v_{Tf}}\right) - g =$$

$$= g\left(1 - \frac{2}{3}\frac{z}{y_f}\right)\left(1 + \frac{z}{y_f}\right)^{-1}\left(1 - 2\sqrt{\frac{6}{\pi}}\frac{u}{v_{Tf}}\right) - g, \qquad \textbf{[0.2 p]}$$

or

$$a \cong g\left(1 - \frac{2}{3}\frac{z}{y_f}\right)\left(1 - \frac{z}{y_f}\right)\left(1 - 2\sqrt{\frac{6}{\pi}}\frac{u}{v_{Tf}}\right) - g \cong g\left(1 - \frac{5}{3}\frac{z}{y_f}\right)\left(1 - 2\sqrt{\frac{6}{\pi}}\frac{u}{v_{Tf}}\right) - g \cong$$

$$\cong g\left(1 - \gamma\frac{z}{y_f} - 2\sqrt{\frac{6}{\pi}}\frac{u}{v_{Tf}}\right) - g = -g\gamma\frac{z}{y_f} - 2g\sqrt{\frac{6}{\pi}}\frac{u}{v_{Tf}} = -\omega_1^2 z - 2\delta u. \qquad \textbf{[0.4 p]}$$

This is the eq. of a damped oscillator, with the natural angular frequency

$$\omega_1 = \sqrt{\frac{\gamma g}{y_f}} \qquad \textbf{[0.1 p]}$$

and the damping coefficient

$$\delta = \sqrt{\frac{6}{\pi}}\frac{g}{v_{Tf}} = g\sqrt{\frac{2\mu}{\pi RT_f}} = \sqrt{\frac{2m_{Ar}g}{\pi m y_f}}. \qquad \textbf{[0.1 p]}$$

In conclusion, the oscillation natural frequency of the piston in its final stage of movement is

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{\gamma g}{y_f}},$$

having the numerical value

$$f_1 = 0.524\ \text{Hz} \cong 0.52\ \text{Hz},$$

while that of the damping coefficient is

$$\delta = 2.58 \cdot 10^{-2}\ \text{s}^{-1} \cong 2.6 \cdot 10^{-2}\ \text{s}^{-1}.$$

The angular pseudofrequency is

$$\omega_1' = \sqrt{\omega_1^2 - \delta^2} = \sqrt{\frac{g}{y_f}\left(\gamma - \frac{2m_{Ar}}{\pi m}\right)},$$ **[0.2 p]**

so, the pseudoperiod is

$$\boxed{T_1' = 2\pi\sqrt{\frac{y_f}{g\left(\gamma - \frac{2m_{Ar}}{\pi m}\right)}}},$$ **[0.2 p]**

having the numerical value

$$T_1' = 1.91\ \text{s} \cong 1.9\ \text{s}.$$ **[0.1 p]**

C5	How long does it take for the piston to stop moving? For this evaluation, express the relaxation time for the piston's oscillations and calculate the numerical value of damped oscillations lifetime, τ_{tot}, knowing that it is considered to be five times the relaxation time. Also, calculate the number N of oscillations effectuated by the piston in this time.	**0.5 points**

Solution:

Since

$$\boxed{\tau = \frac{1}{\delta} = \sqrt{\frac{\pi m y_f}{2m_{Ar} g}}},$$ **[0.2 p]**

Then the oscillations lifetime is

$$\tau_{tot} = 5\tau = 194\ \text{s} \cong 190\ \text{s}.$$ **[0.1 p]**

The number of oscillations is

$$N = \frac{\tau_{tot}}{T_1'} = 101 \cong 100.$$ **[2 x 0.1 p]**

Notes:

1. *For an ideal gas, the mean values for the gas speed v, respectively v^2, in the direction of the piston velocity, are*

$$\langle v \rangle = \sqrt{\frac{2RT}{\pi\mu}},\ \textit{respectively}\ \langle v^2 \rangle = \frac{RT}{\mu}.$$

2. *If needed, the following approximation formula can be used:*

$$(1+x)^n \cong 1 + nx, if\ |x| \ll 1.$$

problem proposed by

Assoc. Prof. Sebastian POPESCU, PhD

Faculty of Physics, Alexandru Ioan Cuza University of Iasi, Romania