Theoretical problem 2: The adiabatic piston

A vertical cylinder with adiabatic walls contains argon, considered as an ideal gas (the molar mass of argon is $\mu = 40\ \mathrm{g/mol}$). The gas is contained in the cylinder with the help of an adiabatic piston, the mass of which is $m = 0.64\ \mathrm{kg}$. The cross-section area of the piston is $A = 80\ \mathrm{cm}^2$. The piston can move without friction inside the cylinder and does not let the gas escape. The air surrounding the cylinder – piston system is evacuated. Initially, the gas temperature is $T_i = 293\ \mathrm{K}$ and the height at which the piston is in static equilibrium, measured as the height of the gas column in the cylinder, is $y_i = 1.0\ \mathrm{m}$. The gravitational acceleration is $g = 9.81\ \mathrm{m/s^2}$ and the universal constant of ideal gases is $R = 8.315\ \mathrm{J\ mol^{-1}\ K^{-1}}$.

A. Gas parameters in the initial state [0.8 points]

A	Derive the mathematical expressions and calculate the numerical values for the gas pressure in the initial equilibrium state and for the mass of argon in the cylinder.	**0.8 points**

B. Very small perturbations [2.3 points]

The purpose of this problem is to find out what happens with the gas and the piston after perturbing the piston. To do this, the piston is suddenly pushed, such that its initial speed u_i is so much smaller than the thermal speed v_{T_i} of the gas in the initial state that their ratio can be neglected.

B1	Derive the mathematical dependence of the gas temperature T of y.	**0.4 points**

B2	Derive the equation describing the position of the piston, with respect to the bottom of the cylinder, as a function of time, $y(t)$.	**1.6 points**

B3	Calculate the entropy variation of the gas in this process.	**0.3 points**

C. Small perturbations [6.9 points]

The speed u of the piston is at least one order of magnitude higher than in the previous part of the problem, but still much smaller than the thermal speed of the gas. Due to the smallness of the piston's speed, there is no turbulence inside the gas. Furthermore, it is found that the pressure is not the same everywhere in the gas volume. When the piston moves upwards, for example, a rarefaction thin layer will appear immediately under the piston. To make a simple model, we will divide the gas volume in two hydrodynamic regions:

- the rarefaction thin layer, situated immediately under the piston, having the following properties:
  - its volume (δV) is negligible when compared to the entire gas volume;
  - the pressure is uniform in the layer, but slightly smaller than in the rest of the gas;
  - the temperature is the same as in the rest of the gas;
- the rest of the gas in which the pressure and the temperature are those of equilibrium.

During the piston movement, its interaction with the gas, mediated by the hydrodynamic layer beneath it, has a macroscopic effect: the appearance of a drag force acting on the piston. In this way the mechanical energy of the piston is degraded into thermal energy and the piston will eventually stop moving.

C1	Derive the expression for the final position of the piston (y_f), measured with respect to the bottom of the cylinder, as well as for the final temperature of the gas (T_f). Calculate their numerical values considering $u_i = 5.0\ \mathrm{m/s}$.	**1.6 points**

C2	Evaluate the variation of entropy in this process ΔS and calculate its numerical value.	**0.8 points**

The strong inequality $u \ll v_T$ means that the integer powers greater than one of the ratio u/v_T can be neglected.

C3	Evaluate the actual pressure p^* in the gas layer beneath the piston during its movement upwards.	**1.6 points**

An exact analysis of the gas and the piston dynamics involves the simultaneous solution of two coupled nonlinear differential equations, a task possible only with the help of a computer. However, the problem can be solved analytically in the asymptotic approximation, that is by analyzing the final stage of the piston movement, when $y = y_f + z$, $z \ll y_f$. The physical significance of y and y_f are those from above. Also, in the final stage of the dynamics, $\frac{mu^2}{m_{Ar}v_{Tf}^2}$ can be neglected because $u \ll u_i$, even if initially, $u_i \lesssim \sqrt{\frac{m_{Ar}}{m}} v_{T_i}$.

C4	Prove that, in the final stage, the movement of the piston is an oscillatory one with damping. Derive the mathematical expression and calculate the numerical value of these oscillations' pseudoperiod.	**2.4 points**

C5	How long does it take for the piston to stop moving? For this evaluation, express the relaxation time for the piston's oscillations and calculate the numerical value of damped oscillations lifetime, τ_{tot}, knowing that it is considered to be five times the relaxation time. Also, calculate the number N of oscillations effectuated by the piston in this time.	**0.5 points**

Notes:

1. *For an ideal gas, the mean values for the gas speed v, respectively v^2, in the direction of the piston velocity, are*

$$\langle v \rangle = \sqrt{\frac{2RT}{\pi\mu}},\ \textit{respectively}\ \langle v^2 \rangle = \frac{RT}{\mu}.$$

2. *If needed, the following approximation formula can be used:*

$$(1+x)^n \cong 1 + nx, if\ |x| \ll 1.$$

problem proposed by

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