Problem 1. *Steiner's symmetry breaking highway*

Symmetries are a guiding principle in modern theoretical physics. Nevertheless, even if a problem exhibits certain symmetries, it is not always the case that all of its solutions can be found using symmetry arguments alone. This phenomenon is known as spontaneous symmetry breaking and is fundamental to understanding the nature of particles in our universe. The problem below is partly meant to exhibit this phenomenon in a much simpler setting.

Mr. Steiner is trying to get into the highway building business. The projects that he is attempting to design and build require that, given a set of cities, he builds a network of highways in such a way that travel between any two cities is possible (*but the path between any two cities does not necessarily have to be the shortest one*). Given this constraint, in order to diminish costs, his goal is to minimize the total length of each highway network that he has to build.

Part I (4 points). For his first batch of projects, Mr. Steiner has to design the highway network interconnecting three cities. In this part, assume that the three cities lie on a homogenous plane with no obstacles.

a) As his first project, he has to find the minimal length highway network connecting three cities which are placed at the tips of an equilateral triangle. Use symmetry arguments to find a design which minimizes the length of the highway network. If the equilateral triangle has edges of length L, find the total length of this highway network.

b) For his next project, he has to find the minimal length highway network connecting three generic points on the plane. In order to do this, he decides to employ vector calculus.

i) Consider two points on the plane, A and O. Show that if the point O is translated by some infinitesimal vector $\vec{\delta}$ (the norm of $\vec{\delta}$ is much smaller than the distance between A and O), then the distance between the points A and O changes by

$$\delta d = \vec{e} \cdot \vec{\delta}$$

where $\vec{e}$ (i.e. $\vec{e} \cdot \vec{e} = 1$) is the unit vector pointing in the direction from A to O.

ii) Use the result from point i) to find the minimal length highway network connecting three points that are the vertices of a triangle whose angles are all smaller than 120 degrees. Specifically, find the shapes for all the roads (straight or curved), the number of intersections and the angles at each intersection between any two edges of the network. Check if your answer from point a) agrees with your prediction.

iii) Find the minimal highway network connecting three points that are the vertices of a triangle that has one angle larger than 120 degrees. Once again, find the shapes for all the roads (straight or curved), the number of intersections, and the intersection angles between any two edges of the network.

Part II (3 points). For his second batch of projects, Mr. Steiner has to design the highway network interconnecting four cities, placed in the vertices of a square. Once again, in this part, assume that the four cities lie on a homogenous plane with no obstacles.

c) Just like in point a), past proposals have used symmetry arguments to find the network which minimizes the length of the highways, this time for a network connecting four cities. Describe such a configuration and determine the total length of the highway network assuming that all the edges of the square have length L. Use the results from part I, to show that the symmetric configuration is the network of minimal length **with a single intersection point**.

d) Mr. Steiner suspects that the symmetric configuration found at point c) is **NOT** the one that minimizes the total length of the network. Use the results from part I to find whether there indeed are other network configurations that minimize the length. If so, find the new minimal length of the network and compare it to the distance found in point c).

e) To verify your result, make a qualitative sketch of the network's overall length as one brings the intersection points for the network found in point d) closer together or further apart along the axis that unites them.

(If the configuration found in part d) contains more than two intersection points, choose to bring only two of them closer together or further apart when drawing the plot. If you found that the configuration in part c) has the minimal length, plot the length of the network as one splits the single intersection point in this configuration into two intersection points.)

Part III (3 points). Mr. Steiner's projects have reached a global scale as the distance between cities has increased to a scale comparable to Earth's radius. Because of that, he has to take Earth's curvature into account when designing the highway network. Assume that Earth is a perfect sphere with radius R with a homogenous surface that has no obstacles. To connect only two cities located on Earth's equator, he has found that the minimal path between the two also follows the equator. Using this fact, help Mr. Steiner design the following two highway networks:

f) Find the minimal highway network connecting a city at the North pole, a city at the South pole and Bucharest, a city whose latitude is 44.42 degrees North and longitude is 26.10 degrees East. To do this, find the shapes for all the roads (straight or curved), the number of intersections, the intersection angles between any two edges of the network, and the total length of the network.

g) Find the minimal highway network connecting a city at the North pole and two cities located on the equator, at a longitude of 30 degrees East, and 30 degrees West, respectively. To do this, find the shapes for all the roads (straight or curved), the number of intersections, and the intersection angles between any two edges of the network.

Hint: Given two points (1 and 2) on the sphere, whose normal vectors to the surface of the sphere are given by $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ then the distance between the two points is given by

$$d = R\theta$$

where θ is the angle between $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$, given by

$$\cos\theta = \overrightarrow{n_1}\cdot\overrightarrow{n_2}$$

Use this to show that the unit vector tangent to the sphere at point 1, pointing from 1 to 2, can be expressed as

$$\overrightarrow{e_{12}} = -\frac{\overrightarrow{n_1}\cdot\overrightarrow{n_2}}{\sqrt{1-(\overrightarrow{n_1}\cdot\overrightarrow{n_2})^2}}\,\overrightarrow{n_1} + \frac{1}{\sqrt{1-(\overrightarrow{n_1}\cdot\overrightarrow{n_2})^2}}\,\overrightarrow{n_2}$$

as long as the vectors $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ are not opposite to each other (i.e. the points are exactly on opposite sides of the sphere).

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