Theoretical problem 3: ***On the propagation and generation of gravitational waves***

The scope of this problem is to help you understand the properties of gravitational waves, whose discovery has recently been rewarded with the 2017 Nobel Prize in Physics. Part I of the problem will analyze the propagation of gravitational waves in close analogy with electromagnetic waves, while Part II will be concerned with estimating the amount of energy gravitational waves carry from the merger of two black holes.

Part I. [6 pts.] Numerous phenomena that occur when studying gravitational waves closely resemble those in the world of electromagnetism. This is due to the similarity between solutions to the Maxwell equations in vacuum,

$$\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0, \tag{1}$$

$$\nabla \cdot \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0, \tag{2}$$

$$\nabla \times \vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{x} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{y} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{z} = -\frac{\partial \vec{B}}{\partial t}, \tag{3}$$

$$\nabla \times \vec{B} = \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)\hat{x} + \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)\hat{y} + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\hat{z} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}, \tag{4}$$

which describe electromagnetic waves propagating freely, and solutions to the equation for gravitational waves propagating in vacuum,

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{h}_{ij} + \nabla^2\mathbf{h}_{ij} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{h}_{ij} + \left(\frac{\partial^2\mathbf{h}_{ij}}{\partial x^2} + \frac{\partial^2\mathbf{h}_{ij}}{\partial y^2} + \frac{\partial^2\mathbf{h}_{ij}}{\partial z^2}\right) = 0 \tag{5}$$

Above, c is the speed of light and $\vec{E}$ and $\vec{B}$ are the electric and magnetic field, respectively, h can be understood as a 3×3 matrix whose elements represent perturbations in the "fabric" of space. The matrix needs to be symmetric ($h_{ij} = h_{ji}$), with each line and column representing a direction in space: specifically, $i, j = \overline{1,3}$, and 1 corresponds to x, 2 to y and 3 to z. Thus, for instance, the element h_{11} is associated to the xx component of the wave, h_{12} is associated to the xy component of the wave etc. The gravitational wave equation (5) thus applies independently to every element of the matrix h.

a) [2 pts.] Show that for an electromagnetic wave propagating in the z-direction, $\vec{E}(\vec{r},t) = E_0 e^{i(\omega t - kz)}\hat{x}$ is a solution to Maxwell's equations in vacuum (1–4), where $\hat{x}$ is the unit vector in the x-direction, and find the associated magnetic field $\vec{B}(\vec{r},t)$. Along the way, prove that $\vec{B}(\vec{r},t)$ needs to be pointing in the y-direction. Furthermore, show that $\vec{k}\times\vec{E}(\vec{r},t)=\omega\vec{B}(\vec{r},t)$.

b) [1 pt.] Prove that the equation for gravitational waves also has a similar oscillating solution propagating in the z-direction, $h=h_0e^{i(\omega t-kz)}$, where h_0 is a 3 × 3 matrix of constants.

c) [0.5 pts.] Show that in vacuum both electromagnetic waves and gravitational waves propagate at the speed of light.

d) [1 pt.] Similar to how Maxwell's equations impose a constraint that relates the electric field, $\vec{E}$, to the magnetic field $\vec{B}$, there are supplementary constrains that can be imposed for the matrix *h*. Specifically, for a wave propagating in the z-direction one can impose that any component of the matrix related to the z-direction has to vanish ($h_{i3}=h_{3i}=0$ for $i=\overline{1,3}$), and that the matrix is traceless ($h_{11}+h_{22}+h_{33}=0$). Use these properties, together with the fact that the matrix is symmetric, to show that h_0 only has two independent components. Show that h_0 can be expressed as

$$\mathbf{h}_0=h_+\epsilon_+ + h_\times\epsilon_\times\,, \tag{6}$$

where h_+ și $h_\times$ are two independent constants and ε_+ and $\varepsilon_\times$ are given by

$$\epsilon_+=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}\,,\qquad \epsilon_\times=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\,. \tag{7}$$

e) [1.5 pts.] In quantum mechanics, the spin of a particle is given by a number *s* that describes the symmetry of the wave solution associated with the propagation of this particle. Specifically, the spin is the maximum number, *s*, for which the wave solution is invariant under rotations of $360^0/s$ around the direction of propagation. The three-dimensional rotation matrix that performs a rotation of angle θ around the z-axis is given by,

$$\mathbf{U}=\begin{pmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{pmatrix}\,, \tag{8}$$

and transforms the electric and magnetic fields as, $\vec{E}\rightarrow U\cdot\vec{E}$, and, $\vec{B}\rightarrow U\cdot\vec{B}$ respectively[1]. The matrix *h* transforms as, $h\rightarrow UhU^T$, where U^T is the transpose[2] of *U*.

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[1] Note that when taking the product between the matrix U and a vector $\vec{E}$, each element of the resulting vector $\vec{E}'$ is given by, $(\vec{E}')_i=\sum_{j=1}^{3}U_{ij}(\vec{E})_j$.

[2] Note that when taking the transpose $(U^T)_{ij}=U_{ji}$ for all $i,j=\overline{1,3}$. When taking the product between two 3×3 matrices, A and B, $(AB)_{ij}=\sum_{l=1}^{3}A_{il}B_{lj}$ for every $i,j=\overline{1,3}$.

Find the spin of the photon (the particle associated to electromagnetic waves) and that of the graviton (associated to gravitational waves).

Part II. [4 pts.] The black holes, whose merger was recently detected by the winners of the 2017 Nobel Prize in Physics, are extremely massive objects with strange physical properties.

The first detection of gravitational waves came from the merger of two black holes each with a mass equal to M = 30 solar masses located $d = 1.3 \cdot 10^{9}$ light-years from Earth (1 solar mass is $1.98 \cdot 10^{30} kg$ and 1 light year is $9.46 \cdot 10^{15} m$).

a) [1.5 pts.] Assuming that the merger of the two black holes occurs while the black holes are close to being at rest and generates only gravitational waves, and nothing else, use the mass-energy relation to express the energy passing through a detector with the opening of 1 m^2, situated on Earth from those gravitational waves. Express your answer both analytically (in terms of the mass M of each black hole, the distance d and the speed of light c) and numerically.

b) [2.5 pts.] The assumption that all the energy from the merger is converted only to gravitational waves can be easily refined using the laws of thermodynamics. Gravitational waves hold only negligible entropy while black holes carry huge amounts of entropy. The entropy of a single black hole of mass M is given by, $S_{BH} = sM^2$, where s is related to fundamental constants. Argue that if two black holes violently collide to make one bigger black hole, then at most 29% of their initial rest energy can be radiated in gravitational waves. Thus, assuming that the two black holes are almost static during the collision, give a numerical upper bound for the energy flux determined on Earth from the black hole collision. Note that the assumption that the two black holes are static should only apply to the numerical result and no to the rest of the problem.

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