Q1. Compressible fluids

The study of gases flow uncovers many interesting phenomena which have a myriad of applications, starting from boilers to airplanes and rockets. To simplify the calculations, in this problem the following assumptions will be adopted:

- The gas is ideal;
- The gas flow is steady and non-turbulent;
- The processes taking place in the flowing gas are adiabatic;
- The gas flow speed is much less than the speed of light;
- The gas flow is uniform and one-dimensional (axisymmetric);
- The effect of gravity is negligible.

The constants useful in this problem are:

- the molar mass of air, $\mu = 29.0\ \mathrm{g/mol}$;
- the ideal gas constant, $R = 8.32\ \frac{\mathrm{J}}{\mathrm{mol\cdot K}}$.

A. Bernoulli's equation

Bernoulli's equation is the mathematical form of the law of conservation and transformation of energy for a flowing ideal gas. It bears the name of the Swiss physicist Daniel Bernoulli (1700 - 1782), who derived it in 1738. The easiest way to obtain this equation is to follow a fluid particle (a volume element of the fluid) in its way on a streamline.

A	Perform the energy balance between two points in the flowing gas, knowing the parameters (p_1, ρ_1, v_1) and (p_2, ρ_2, v_2), and derive the equation that connects these variables. The adiabatic exponent γ of the gas is also known. The parameter p is the gas pressure, ρ its density, and v its speed.	1.5 p

B. Propagation of a perturbation in a flowing gas

If the pressure in a layer of a macroscopically motionless gas system suddenly increases (by heating or rapid compression), the layer will begin to expand, compressing the adjacent layers. This pressure disturbance will be thus transmitted by contiguity as an elastic wave through the gas.

B1. Speed of the perturbation

The speed c of this wave is the speed of its wavefront (the most advanced surface, the points of which oscillate in phase and the thermodynamic parameters of which have the same value). If in the reference frame of the unperturbed gas the process of the wave propagation is nonsteady (the gas parameters in any point vary with time), in the reference frame of the wavefront the process will be steady, so the simple equations for a steady state can be applied.

B1	Derive the mathematical expression for the speed c of the wavefront, taking into account that the thermodynamic parameters of the unperturbed gas are (p, ρ), while those "behind" the wavefront are $(p_1, \rho_1) = (p + \Delta p, \rho + \Delta\rho)$.	1.5 p

B.2 Sound waves

Sound waves are waves of weak disturbances ($\Delta p \ll p$ and $\Delta\rho \ll \rho$) that travel fast enough, their speed being of the order of hundreds of meters per second. Due to this, the gas compressions and rarefactions can be considered as adiabatic, the adiabatic exponent being γ.

B2	Using the result from **B1**, obtain the mathematical expression for the sound speed in the gas and, using Bernoulli's eq., derive the relation between the flow speed at a given point in the gas and the local sound speed.	0.5 p

B.3 Mach's number

For classifying the speed performances of bodies in a fluid (*e.g.* aircrafts), as well as the flow regimes of fluids, the Swiss aeronautical engineer Jakob Ackeret (1898 – 1981) – one of the leading authorities in the 20th century aeronautics – proposed in 1929 that the ratio of the body or of the fluid's speed v and the local sound speed c in that fluid to be called Mach's number

$$M = \frac{v}{c},$$

after the name of the great Czech (then in the Austrian empire) physicist and philosopher Ernst Mach (1838 – 1916). Primarily, the value of this non-dimensional quantity delimitates the incompressible from the compressible behavior of a flowing fluid, in aeronautics this limit being settled to $M = 0.3$.

B3.1	Find the relative variation of the gas density as a function of Mach's number, when its motion is slowed down to a stop, its initial velocity being $v < c$, and calculate its maximum value for a flow to be considered incompressible.	0.5 p

B3.2	The pressure at the nose of an aircraft in flight was found to be $1.92 \cdot 10^5$ Pa and the speed of air relative to the aircraft was zero at this point. The pressure and temperature of the undisturbed air were $1.01 \cdot 10^5$ Pa and 21.1 °C respectively. The adiabatic exponent for this temperature is $\gamma = 1.40$. Find the speed and the Mach number of the aircraft.	0.5 p

B3.3	When a gas is flowing through a pipe, it exerts a friction force on the fluid, which is not always negligible. If at the entrance of such a pipe the static pressure in the flowing fluid is $p_1 = 6.90 \cdot 10^5$ Pa and the Mach number is $M_1 = 0.700$, while at the exit $M_2 = 1.00$, find the expression and the numerical value of the force with which the fluid is acting on the pipe. The adiabatic exponent is $\gamma = 1.40$, the constant cross section of the pipe is $S = 9.29 \cdot 10^{-2}$ m^2, and the relative increase of the gas temperature through the pipe is $5.00 \cdot 10^{-3}$.	1.0 p

C. Shock waves

There are two types of acoustic waves in a gas: sound waves and shock waves. The latter appear when a body moves in a gas with a supersonic speed (*i.e.* the relative speed of the body with respect to that of the gas is greater than the sound speed). At supersonic speeds, in front of the body appears a very thin layer of gas with a higher pressure, called *compression shock*. This kind of special acoustic waves were studied by Mach, so the envelope of such a wave is known as Mach's cone, having the body in its apex. Passing through the compression shock, the thermodynamic parameters of the gas change abruptly. The Mach's cone is an example of an oblique shock, but we are interested here mainly in normal shocks, for which the shock wavefront is perpendicular on the body or fluid velocity.

For shock waves the pressure/density differences between the two sides of the wavefront can reach very high values. Passing through the wavefront, the thermodynamic parameters vary abruptly, with a sudden jump. This is another reason for which a shock wavefront is called a pressure or a compression shock.

C.1 The shock adiabat

The gas compressed by the shock wave undergoes an irreversible adiabatic process which cannot be described by Poisson's equation. However, an equation for the shock adiabat was deduced towards the end of the XIXth century by the Scottish physicist William Rankine (1820 – 1872) and, independently by him, by the French engineer Pierre Henri Hugoniot (1851 – 1887), using the mass and energy conservation, as well as the momentum equation. The Rankine – Hugoniot equation, or the *shock adiabat*, relates the pressure and the density of the gas compressed by a shock wave.

C1	Denoting with p_s and ρ_s the gas pressure and density in front of the compression shock (which are known), and with p_1 and ρ_1 the same parameters behind the shock (which are unknown), show that the pressure ratio $\frac{p_1}{p_s} = y_1$ is related with the density ratio $\frac{\rho_1}{\rho_s} = x_1$ by a relation of the form $$y_1 = \frac{\alpha x_1 - \beta}{\tau - \sigma x_1}.$$ Find the explicit form of the coefficients α, β, τ and σ. The adiabatic exponent γ of the gas is known. <u>Note</u>: *For simplicity, use a stream tube with a constant cross section, crossing perpendicularly the wavefront of the normal shock.*	1.5 p

C.2 A shockwave created by an explosion

An explosion creates a spherically shockwave propagating radially into still air at $p_s = 1.01 \cdot 10^5$ Pa and $t_s = 20.0$ °C. A recording instrument registers a maximum pressure of $p_1 = 1.48 \cdot 10^6$ Pa as the shock wavefront passes by. The adiabatic coefficient of air for this compression shock is $\gamma = 1.38$, the molar mass of air is $\mu = 29.0 \frac{\mathrm{g}}{\mathrm{mol}}$ and the ideal gas constant is $R = 8.32 \frac{\mathrm{J}}{\mathrm{mol \cdot K}}$.

C2.1	Determine the air temperature increase $\frac{T_1}{T_s}$ under the action of the compression shock.	0.5 p

C2.2	Determine the Mach's number corresponding to the speed of the shockwave.	0.5 p

C2.3	Determine the wind's speed v_1 following the shock wavefront, with respect to a fixed observer.	0.5 p

During the compression shock the gas temperature and pressure sharply increase, much more than in a quasistatic adiabatic compression. After the shock, the gas expands adiabatically, but because the slope of the adiabatic process is smaller than that of the adiabatic shock, when the gas density reaches again the initial value, its pressure p_2 is still higher than that of the unperturbed gas, p_s.

C2.4	Derive the ratios $\frac{p_2}{p_s}$ and $\frac{T_2}{T_s}$ at the end of the expansion process and calculate the numerical values of p_2 and T_2.	0.5 p

C2.5	From this point the gas is cooling until it reaches the initial state. Assuming that for the entire cyclic process the adiabatic exponent has the same value, derive an expression for the entropy variation of the mass unit of air during the compression shock and calculate its numerical value.	1.0 p

© Assoc. Prof. Sebastian POPESCU, PhD
Faculty of Physics, Alexandru Ioan Cuza University of Iași, ROMANIA

Q1. Compressible fluids

Answer sheet

The equation that relates (p_1, ρ_1, v_1) and (p_2, ρ_2, v_2) is

A	

B1	$c =$

B2	$c =$ The relation between the flow speed at a given point in the gas and the local sound speed is:

B3.1	$\frac{\Delta\rho}{\rho}(M) =$ $\left(\frac{\Delta\rho}{\rho}\right)_{max} =$	
B3.2	$v =$ $M =$	
B3.3	The expression of the force is: $F =$	The numerical value of the force is: $F =$
C1	$\alpha =$ $\beta =$	$\tau =$ $\sigma =$
C2.1	$\frac{T_1}{T_s} =$	

C2.2	$M =$

C2.3	$v_1 =$

	Analytical expressions	Numerical values
C2.4	$\frac{p_2}{p_s} =$	$p_2 =$
	$\frac{T_2}{T_s} =$	$T_2 =$

	Analytical expression	Numerical value
C2.5	$\frac{\Delta S_{shock}}{\Delta m} =$	$\frac{\Delta S_{shock}}{\Delta m} =$

Q2. Sources, atoms, and spectra

A. Light source

In its proper reference frame, a point source emits light in the form of a divergent conical beam, with the angular width of 90° (from -45° to +45° with respect to the cone axis). In a reference frame which moves towards the source with an unknown speed v, the angular width of the beam is of only 60° (from -30° to +30° with respect to the same cone axis). The light speed in vacuum is $c = 2.998 \cdot 10^8\ \frac{\mathrm{m}}{\mathrm{s}}$.

A	Determine the speed v of the source.	2.50 p.

B. Balmer emission spectrum

The spectral resolving power of a spectrometer is $R = 5 \cdot 10^5$. The spectrometer is used to observe the Balmer series in the emission spectrum of the hydrogen atom (the visible domain).

Note: The possible mechanisms of broadening of the spectral lines (Lorentzian, Gaussian, etc.) will not be considered.

B.1	Express the mathematical definition of the spectral resolving power of the instrument.	0.25 p.

B.2	Determine the highest value for the principal quantum number n of the energy level for which the spectral line emitted by an atom for the transition to the level $n' = 2$ can still be distinctly resolved by the instrument, with respect with its neighbours.	2.25 p.

C. Absorption spectra

The energy levels of an atom are given by $E_n = -\frac{A}{n^2}$, where n is an integer and A is a positive constant. Among the adjacent spectral lines which, at room temperature, the atom can absorb, two have the wavelengths 97.5 nm and 102.8 nm , respectively. The elementary electric charge is $e = 1.602 \cdot 10^{-19}$ C, the speed of light in vacuum is c= $2.998 \cdot 10^8\ \frac{\mathrm{m}}{\mathrm{s}}$, and Planck's constant is $h = 6.626 \cdot 10^{-34}$ J·s.

C.1	Find the values of the quantum numbers n of the energy levels implied in the transitions.	3.00 p.

C.2	Determine the value of the constant A in joule and in electron-volt.	1.50 p.

C.3	Identify the nature of the atom and justify the choice made.	0.50 p.

Answer sheet

A	Final expression for the speed of the source	Numerical value for the speed of the source	
	$v =$	$v =$	2.50 p.

B.1	$R =$	0.25 p.

B.2	$n =$	2.25 p.

C.1	$n =$	3.00 p.

C.2	in joule	$A =$	1.00 p.
	in electron-volt	$A =$	0.50 p

C.3		0.50 p.

Theoretical Problem No. 3 (10 points)

"Squeezing" electrical charge carriers using magnetic fields

Plasma physics has to solve the problem of achieving devices capable of producing energy on a large scale through fusion. There are no issues related to the feasibility of the scientific method - the process can be observed as it happens in stars. But there are many technological feasibility problems primarily related to heating the plasma, and controlling it, at temperatures like the ones in the Sun. Physical conditions of the nuclear fusion cannot be achieved in containers. Maintaining plasma localized in limited volume can be accomplished using magnetic fields.

The first two tasks of the problem require analyzing several situations in which the movement of charged particles is limited by external magnetic fields. The third task asks you to study the confinement of electrical charge carriers through their own magnetic field. When solving the problem you may rely on the following numerical values: elementary electric charge $e = 1.6 \times 10^{-19} C$, mass of the electron $m = 9.1 \times 10^{-31} kg$, the magnetic permeability of vacuum $\mu_0 = 4\pi \times 10^{-7} F \cdot m^{-1}$.

Task No. 1

A very long metallic cylinder (a perfect electric conductor), having the length L and the radius R, $(L >> R)$ rotates at constant angular speed ω around its axis of symmetry.

The cylinder is located in a homogeneous magnetic field whose induction $\vec{B}$ is parallel to the axis of symmetry of the cylinder. The mass of electron is m and his electric charge is $-e$. Let the dielectric permittivity of the material from which the cylinder is made be ε.

1.i. For a stationary situation determine the expression of the bulk density of electric charge ρ into the cylinder at a distance r from its axis of symmetry $(0 < r < R)$. Express the result as function of B, e, m, ω and ε. *(1.00p)*

1.ii. Determine the expression of angular velocity ω_0 so that the bulk charge density is zero at any point of the cylinder. Express the result as function of B, e and m. *(0.20p)*

1.iii. Evaluate the possibility of the practical realization of a zero bulk charge density in any point of the metallic cylinder, when an experiment of the type described in question is carried out in the Earth's magnetic field in a place where the magnetic field induction is $B = 1{,}82 \times 10^{-5} T$. Briefly argue the answer. *(0.30p)*

Task No. 2

In a vacuum chamber there is a long, straight wire of negligible thickness made from a material with high electrical conductivity. An electrical current with the intensity of $I = 10A$ passes through the wire. Electrons are sent on a direction perpendicular on the wire, towards the wire. Their motion starts at the distance r_0 from the wire, with an initial speed v_0 much smaller than speed of light. The electrons cannot approach the wire at a distance less than $r_0/2$. Consider two reference frames – one being the laboratory system S.L. and the other being a mobile system S.M. that runs parallel to the wire with a constant speed v_0 in the direction in which the current flows through the wire.
Neglect the magnetic field of Earth.

2.i. Deduce the expressions of induction of the magnetic field produced by the current flowing through the wire in each of the two reference frames $\vec{B}_{S.L.}(r), \vec{B}_{S.M.}(r)$. *(1.00p)*

2.ii. Determine the expression of the difference between Lorentz forces in the two reference systems, $\vec{F}_{S.L.}(r)-\vec{F}_{S.M.}(r)$, forces acting on the electron sent to the wire. Express the result in terms of e, μ_0, I, r and v_0. *(1.50p)*

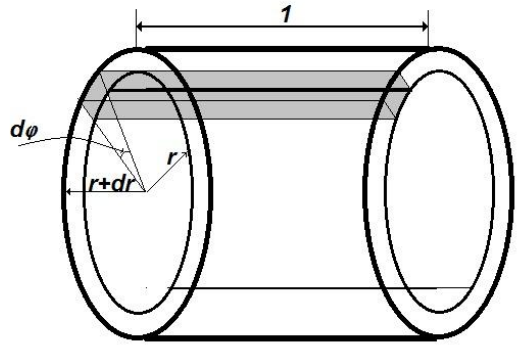
2.iii. State the values of the electron velocity's $\vec{v}(r_0/2)$ components in both reference frames. *(1.00p)*

2.iv. Determine the expression of the speed v_0 as function of e, m, I, μ_0. Calculate the numerical value of v_0. *(1.50p)*

2.v. Deduce the expression of the maximum distance from the wire D at which one can find the electron, as function of r_0, when the electron moves away from wire on a direction perpendicular on it. *(0.50p)*

Task No. 3

A cylindrical column of plasma having the radius R and the length L is generated in a vacuum chamber. The plasma appears as result of ionization of a gas, such that the concentrations of electrons $n_e(r)$ and ions $n_i(r)$ are equal at every point $n_i(r)=n_e(r)=n(r)$; the common value $n(r)$ depends only on the distance between the point and the axis of symmetry of the plasma cylinder. It is assumed that the plasma is in a stationary state, and therefore all its macroscopic characteristics are independent on time. It may be admitted that the temperature T of plasma is the same at every point of the column and that at these temperature the parameters describing plasma abide the perfect gas law. The electric charge of electron is $-e$ and the ions are monovalent, carrying an electric charge e. Consider as known the magnetic permeability of plasma μ and Boltzmann's constant k_B. Between the electrodes at the ends of the plasma column passes through the plasma an electrical current characterized by a constant density $j(r)=j$.



Consider an elementary portion of the hollow cylinder having the radiuses r and $r+dr$ as in joined figure. Elementary portion has a height equal to the unit. In the annulus (circular crown) representing the cross section of the plasma column the elementary portion covers the angle $d\varphi$.

3.i. Write the expressions of the forces acting on this elementary volume of plasma ($\vec{F}_p$ due to pressure $p(r)$ in the column of plasma, $\vec{F}_e$ due to the interaction with electrical charges carriers that there are in plasma , $\vec{F}_m$ due to the interaction with magnetic field of electrical current flowing through plasma). Write the equation describing the equilibrium of the considered elementary portion of plasma. *(1.20p)*

3.ii. Deduce the expression of pressure $p(r)$ in a point of the column of plasma. Express the answer as function of r, j, R and magnetic permeability μ. *(1.00p)*

3.iii. Determine the expression of the number of particles N carrying electrical charge in the column of plasma as function of L, I, T, μ și k_B. *(0.80p)*

© Topic proposed by:
Dr. Delia DAVIDESCU
Dr. Adrian DAFINEI

ANSWER SHEET

Theoretical Problem No. 3 (10 points)

"Squeezing" electrical charge carriers using magnetic fields

Task No. 1

1.i. The expression of the bulk density of electric charge ρ into the cylinder at a distance r from its axis of symmetry		1.00p
1.ii. The expression of angular velocity ω_0 so that the bulk charge density is zero at any point of the cylinder		0.20p
1.iii. Evaluate the possibility of the practical realization of a zero bulk charge density in any point of the metallic cylinder. Briefly argue the answer		0.30p

Task No. 2

2.i. The expressions of induction of the magnetic field produced by the current flowing through the wire in each of the two reference frames $\vec{B}_{S.L.}(r)$ and $\vec{B}_{S.M.}(r)$		1.00p
2.ii. The expression of the difference between Lorentz forces in the two reference systems, $\vec{F}_{S.L.}(r)-\vec{F}_{S.M.}(r)$		1.50p

2.iii. State the values of the electron velocity's $\vec{v}(r_0/2)$ components in both reference frames	1.00p

2.iv. The expression of the speed v_0 as function of e, m, I, μ_0 and the numerical value of v_0		1.50p
2.v. The expression of the maximum distance from the wire D at which one can find the electron, as function of r_0, when the electron moves away from wire on a direction perpendicular on it		0.50p

Task No. 3

3.i. The expressions of the forces acting on this elementary volume of plasma		0.80p
The equation describing the equilibrium of the considered elementary portion of plasma		0.40p
3.ii. The expression of pressure $p(r)$ in a point of the column of plasma		1.00p

3.iii. The expression of the number of particles N carrying electrical charge in the column of plasma		0.80p