# PROBLEM 1: CIRCLES ALL THE WAY

*This problem has two parts.*

## A. Determining the radius of curvature of a planar curve by means of Mechanics

Each infinitesimal length element of a curve can be thought as belonging to a circle with a certain radius (see **Figure 1**). This radius is called the *radius of curvature*. In order to determine the radius of curvature at some point of a planar curve, one can consider that the curve is the trajectory of a point-like object, and that the given point is the very object.

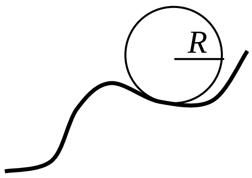


**Figure 1**

Let $y = f(x)$ be the equation of the curve, where $x$ and $y$ are the coordinates of the object.

**a.** Express the components $v_x$ and $v_y$ of the velocity of the object in terms of $x$ and its time derivatives, and $f(x)$ and its derivatives.
**b.** Let $a_t$ be the component of the acceleration vector parallel to the velocity. Express vector $\vec{a}_t$ in terms of $v_x$, $v_y$, the components $a_x$ and $a_y$ of the acceleration, and the unit vectors $\vec{i}$ and $\vec{j}$ of the O$x$ and O$y$ axes.
**c.** Express the magnitude (i.e. the absolute value) of the component $\vec{a}_n$ of the acceleration perpendicular to the velocity, in terms of $v_x$, $v_y$, $a_x$ and $a_y$.
**d.** Express the radius of curvature in terms of $f(x)$ and its derivatives.
**e.** Find the radius of curvature of the parabola $y = Ax^2$ ($A > 0$) at a point having coordinate $x = x_0$.
**f.** Find the period of small oscillations performed by a bead on the bottom of a smooth surface of equation $y = \sin 2x$ [m]. You may use $g \approx \pi^2$ [m/s$^2$], where $g$ is the gravitational acceleration.

## B. Springs on a circle

In this problem we will investigate the motion of a point-like object of mass $m$ connected to $N + 1$ springs of stiffness $k$. The springs are attached to a circle of radius $R$ (see **Figure 2**) and have negligible natural length. The points to which the springs are attached are arranged uniformly on the circle such that, when the object is at the center of the circle, the angle between two adjacent springs is equal to $2\pi/(N+1)$. We label the springs 0 through $N$.

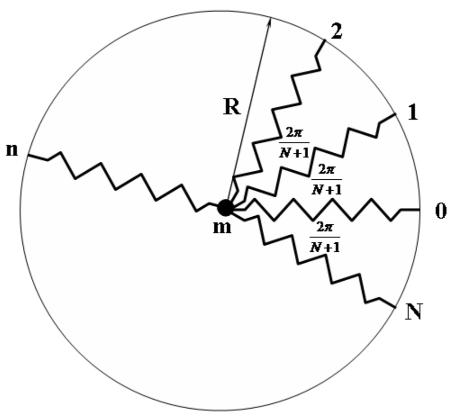


**Figure 2**

Let angle α and the radial coordinate $r$ of the object be defined as in **Figure 3**. The object can move in the plane of the circle and we ignore the effects of gravity throughout the problem.

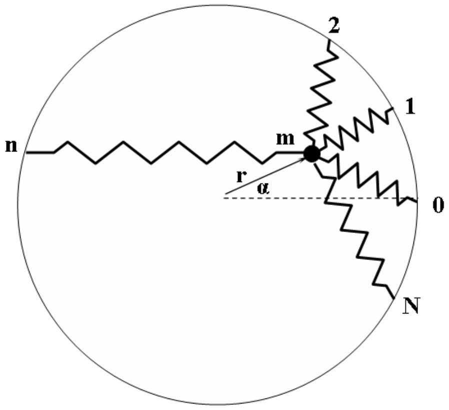


**Figure 3**

**g.** Compute the length $l_n$ of the $n$-th spring for arbitrary $r$ and α.

**h.** Write down the kinetic energy $E_{\text{kin}}$ and the potential energy $E_{\text{pot}}$ of the object in terms of $r$, α, and their time derivatives. (Do not evaluate the sum in the potential energy just yet.)

**i.** Compute the sums

$$\sum_{n=0}^{N} \cos\left(\frac{2n\pi}{N+1}\right) \quad \text{and} \quad \sum_{n=0}^{N} \sin\left(\frac{2n\pi}{N+1}\right)$$

and use the results to evaluate $E_{\text{pot}}$ for arbitrary $N$.

**j.** Show that the angular momentum $L$ is conserved.

**k.** Write down the implicit equation for $r$ in terms of $\mathcal{L} \equiv L/m$ and $\omega^2 \equiv (N+1)k/m$.

**l.** Perform the substitution

$$r(t) = \sqrt{z^2(t) + K}\,,$$

with $K$ an unspecified parameter, and obtain the equation "of motion" for $z(t)$.

Perhaps surprisingly, this equation admits oscillatory solutions $z(t) = A\cos(\omega t + \phi_0)$.

**m.** Show this, and determine $A$ in terms of $\mathcal{L}$, $K$, and ω.

$K$ and $\phi_0$ can be thought of as integration constants to be specified by the initial conditions. Since the equation of motion is second order, and we have two integration constants, it means that this is the most general function describing the distance $r(t)$. On the other hand, one can notice that the equation for $r(t)$ remains unchanged on substituting $r$ with $-r$. Physically this is the same as substituting α with α + π, which means that the distance from the center of the circle to the object is the same in any two opposing directions.

**n.** Write down $r(t)$ in terms of $\mathcal{L}$, ω, $K$ and $\phi_0$. Is $r(t)$ periodic? Is the motion periodic? Do they both have the same period?

**o.** Describe the motion of the object when $L = 0$.

**p.** Find out the possible value of $r$ at which the object could perform circular uniform motion.

Suppose now we remove the springs labeled 0, $d$, $2d$, … , where $d$ divides $N + 1$.

**q.** Argue that $r(t)$ derived in part **n.** continues to hold, but for a different value of ω. What is this new value ω' in terms of ω, $N$, and $d$?

# Theoretical Problem no. 2 (10 points)

## Terrestrial climate modeling

*Climate change and global warming, the life of humans in new climatic conditions becomes topics of public interest in recent decades.*

*On a cosmic scale, the only phenomena occurring in Earth's energy balance are absorption and emission of radiation. One can say that the earth's climate condition depends on delicate balance between the energy that our planet receives for the formidable energy source that is the sun and the energy that Earth radiates into space. Consider that solar constant for the radiation coming from the Sun towards the Earth is* $w_S = 1370\ W \cdot m^{-2}$.

*A body that absorbs electromagnetic radiation reaching the surface, regardless of the wavelength of the radiation and emits electromagnetic radiation according to the temperature of its surface is called a blackbody. A black body emits energy with a specific spectral distribution, depending on its own temperature. Solid smooth lines, in the graph in Figure 1 show the spectral distribution of the emission energy of the Sun, equivalent to a black body.*

*If* $\varepsilon(T)$ *represents the total energy (in the whole spectrum) emitted by unit area of a black body in unit time, and* $T$ *is the absolute temperature of the black body, then the Stefan - Boltzmann states that* $\varepsilon(T) = \sigma \cdot T^4$. *In the expression* $\sigma = 5{,}67 \cdot 10^{-8}\ W \cdot m^{-2} \cdot K^{-4}$ *is Stefan – Boltzmann constant.*

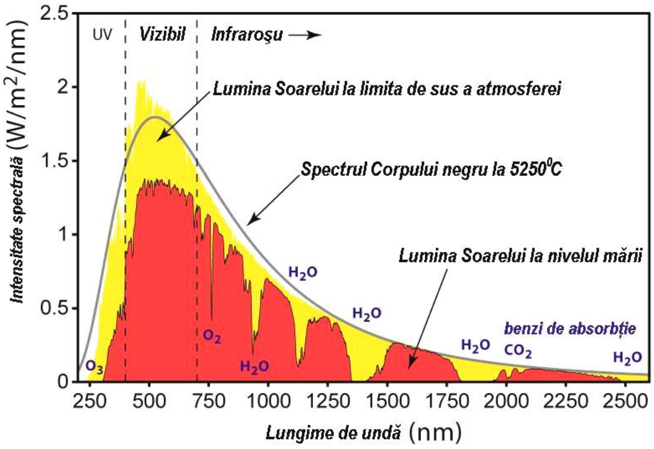


*Figure 1 (not for calculations purposes)*

Problem proposes you to use different modeling in different workloads and to determine the average temperature the Earth would have at the surface for each of the models used. Express the answers as function of symbols of quantities or numerical values marked in blue color in the problem statement. Express obtained numerical results in the form of integers.

## Task no. 1

In the task no. 1 uses simple modeling. Suppose that the energy is taken from the Sun and that the Earth loses energy as a black body.

**1.a.** Determine the expression of average temperature $T_P$ that the Earth's surface should have , in accordance with simple modeling used. (0,70p)

**1.b.** Calculate the numerical value $T_P$ of the average temperature that the Earth's surface should have, according to this model. (0,30p)

## Task no. 2

Simple modeling of the task no. 1 one assumes that the Earth is a black body. Assumption is unrealistic, because all images taken from space show the Earth as a luminous body.

Earth's atmosphere (especially clouds) reflects approximately 24% of the energy coming from the Sun and the Earth's surface (especially ice areas) still reflects 6% of the incident energy. Feature called albedo measures the ratio of reflected radiant flux and incident radiant flux. Considers that terrestrial albedo is $A = 30\%$.

**2.a.** Determine the expression of average temperature $T_P'$ that the Earth's surface should have, in accordance with simple modeling used in these workload. (0,70p)

**2.b.** Calculate the numerical value $T_P'$ of the average temperature that the Earth's surface should have, according to the model in workload 2a. (0,30p)

## *Task no. 3*

While the Sun emits energy mostly in the visible, relatively low temperature of the Earth's land surface causes its emission to be localized in the infrared spectrum. A careful modeling of climate takes account of the atmosphere. The atmospheric gas is a mixture having specific absorbent properties. The Spectrum illustrated in Figure 1 shows that the different atmospheric gases absorb radiation in different spectral domains. Characterization of the spectral absorption can be achieved by introduction of transmission coefficients in the visible $\alpha_{vis}$ and infrared $\alpha_{ir}$, respectively, representing the ratio of the energy passing through the air (in the visible or infrared), and the energy that enters the atmosphere. If radiation in a specified spectral range is completely absorbed, then the corresponding transmission coefficient is zero, and if radiation is not absorbed at all, then the transmission coefficient is one.

Task no. 3 proposes to determine the Earth's temperature $T_P''$ using a modeling that takes into account the partial reflection of light coming from the sun (albedo $A$) and absorptive properties of the atmosphere (transmission coefficients $\alpha_{vis}$ and $\alpha_{ir}$).

**3.a.** Determine the expression of average temperature $T_P''$ that the Earth's surface should have, in accordance with modeling used in these workload. (2,50p)

**3.b.** Calculate the numerical value $T_P''$ of the average temperature that the Earth's surface should have when $A = 0{,}3$, $\alpha_{vis} = 0{,}8$ and $\alpha_{ir} = 0{,}1$. (0,30p)

**3.c.** Using modeling proposed in this workload, calculate the average temperature on the Earth surface, where albedo and transmission coefficients have the values given in Table 1 (1,20p)

*Table 1*

| Case | I | II | III | IV |
|---|---|---|---|---|
| $\alpha_{vis}$ | 1 | 1 | 1 | 1 |
| $\alpha_{ir}$ | 1 | 1 | 0 | 0 |
| $A$ | 0,3 | 0,0 | 0,0 | 0,3 |
| $T''_P\,(K)$ | | | | |
| $t''_P\,(°C)$ | | | | |

## *Task no. 4*

Use modeling proposed in the task no. 3 and assume that the distance between Earth and the Sun would rise by $f = 1\%$.

**4.a.** Determine the expression of average temperature $T_P'''$ that the Earth's surface should have, if $A = 0{,}3$, $\alpha_{ir} = 0{,}3$ and $\alpha_{vis} = 0{,}6$. (1,00p)

## *Task no. 5*

Assume that some of the sand found in the Sahara desert would be "transformed" into a "mirror glass".

**5.a.** Using the modeling proposed in task no. 3 estimates the surface that "mirror glass" should have so that the average temperature at the Earth's surface to fall $1°C$ against the value determined in the task 3.b. Consider that the radius of the Earth is $R_P = 6400\,km$. (3,00p)

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# *Answer Sheet*

## *Theoretical Problem no. 2 (10 points)*

## *Terrestrial climate modeling*

### *Task no. 1*

| | | |
|---|---|---|
| **1.a.** Expression of average temperature $T_P$ that the Earth's surface should have , in accordance with simple modeling used | | 0,70p |
| **1.b.** Numerical value $T_P$ of the average temperature | | 0,30p |

### *Task no. 2*

| | | |
|---|---|---|
| **2.a.** Expression of average temperature $T_P'$ of the Earth's surface | | 0,70p |
| **2.b.** Numerical value $T_P'$ of the average temperature | | 0,30p |

### *Task no 3*

| | | |
|---|---|---|
| **3.a.** Expression of average temperature $T_P''$ of the Earth's surface | | 2,50p |
| **3.b.** Numerical value $T_P''$ of the average temperature | | 0,30p |

**3.c.** The average temperature on the Earth surface, where albedo and transmission coefficients have the values given in Table 1

*Table 1*

| Case | I | II | III | IV |
|---|---|---|---|---|
| $\alpha_{vis}$ | 1 | 1 | 1 | 1 |
| $\alpha_{ir}$ | 1 | 1 | 0 | 0 |
| $A$ | 0,3 | 0,0 | 0,0 | 0,3 |
| $T''_P\,(K)$ | | | | |
| $t''_P\,(^\circ C)$ | | | | |

1,20p

## Task no. 4

**4.a.** Value of average temperature $T_P'''$ that the Earth's surface should have if $A = 0{,}3$, $\alpha_{ir} = 0{,}3$ and $\alpha_{vis} = 0{,}6$.

1,00p

## Task no. 5

**5.a.** The estimated value of surface of mirror glass

3,00p

# PROBLEM 3: BLACK HOLES PHYSICS

*In this problem we will explore the physics of black holes – astrophysical objects so massive that no object, not even a photon, can escape from if it gets sufficiently close. Because black holes are extremely massive, in their vicinity Newton's theory of gravity breaks down and one is forced to use Einstein's general theory of relativity to obtain a correct description of their physics.*

Any black hole in the physical universe is uniquely specified by exactly three quantities: its mass $M$, angular momentum $J$, and charge $Q$. In addition to these, a black hole also has a *space-time singularity* and an *event horizon*, which is the surface surrounding the central singularity which can only be crossed "going in". Any photon or object which falls through the event horizon will not be able to exit back out and will eventually hit the central singularity.

It is convenient to describe the spacetime of non-rotating black holes by four coordinates: $t$, $r$, $\theta$ and $\phi$, with $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$. These can be thought of as the usual spherical coordinates plus time. Because the geometry is no longer flat, the infinitesimal spacetime interval is given by

$$(ds)^2 = -c^2 f\,(dt)^2 + \frac{(dr)^2}{g} + r^2 (d\theta)^2 + r^2 \sin^2\theta\,(d\phi)^2 \qquad (1)$$

with $f$ and $g$ functions of the four coordinates. For spherically symmetric uncharged black holes (known as *Schwarzschild black holes*), $f$ and $g$ are functions of the radial coordinate $r$ only, and are given by

$$f(r) = g(r) = 1 - \frac{r_S}{r}, \qquad (2)$$

where $r_S$, the *Schwarzschild radius*, is the radial coordinate of the event horizon.

**a.** Write down the infinitesimal spacetime interval $(ds)^2$ for Minkowski spacetime in spherical coordinates, using the signature given above (–, +, +, +).

Suppose an observer is initially at rest ($dr/dt = 0$, $d\theta/dt = d\phi/dt = 0$) at radial coordinate $r_0 > r_S$. Under the gravitational pull of the black hole he will start falling towards the event horizon, moving only along the radial direction (and thus keeping $\theta$ and $\phi$ constant at all times). Let $t'$ be the proper time measured by the observer's clock. The correct general relativistic relation between elapsed proper time $dt'$, elapsed coordinate time $dt$ and radial coordinate change $dr$ is (you don't have to show this)

$$\frac{d^2 r}{dt'^2} + \frac{r_S c^2}{2r^2}\left(1 - \frac{r_S}{r}\right)\left(\frac{dt}{dt'}\right)^2 - \frac{r_S}{2r^2}\frac{1}{1 - \frac{r_S}{r}}\left(\frac{dr}{dt'}\right)^2 = 0. \qquad (3)$$

**b.** From equations (1) - (3) compute the proper acceleration $a \equiv d^2r/dt'^2$ in terms of $c$, the speed of light in empty space, $r$, and $r_S$.

***Hint:*** The spacetime interval $(ds)^2$ is the same in all reference frames.

**c.** Even without resorting to any calculation, it should be expected that the Newtonian expression for $a$ can be recovered in the large $r$ (i.e. small $M$) limit of the relativistic

expression. However, now that you have done the calculation, can you make a stronger statement? Determine $r_S$ in terms of $M$, the black hole's mass, $c$, and $G$, the gravitational constant.
**d.** Compute the proper velocity $v \equiv dr/dt'$ as a function of $r$, in terms of $M$ and $r_0$ and $G$.
**e.** Using that

$$\int \sqrt{\frac{x}{1-x}}dx = -\sqrt{x(1-x)} - \arccos\left(\sqrt{x}\right) + C, \tag{4}$$

compute the proper time $t'$ after which the observer reaches the event horizon. Write the result in terms of $r_0$, $M$, $G$ and $c$.

The results of parts **a.** - **c.** may lead one to speculate that general relativity is not that different from Newtonian mechanics after all. This is a misleading interpretation, as the two theories differ significantly in many aspects.
**f.** To highlight one such aspect, calculate the coordinate time $t$ it takes for the observer to reach the event horizon. Are you surprised?
***Hint:*** You do not need to determine the antiderivative in order to compute what the integral equals.

We now turn towards the thermodynamical properties of black holes. If only classical physics is taken into account, black holes do not emit any form of radiation and can thus be considered to have zero temperature. However, in 1974 physicist Stephen Hawking proved that once quantum corrections are considered, black holes emit radiation according to the blackbody spectrum (you do not need to show this). The corresponding blackbody temperature is known as the *Hawking temperature*, $T_H$, and can be thought of as the temperature of the black hole. For a Schwarzschild black hole of mass $M$, the Hawking temperature is equal to

$$T_H = \frac{\hbar c^3}{8\pi G k_B M}, \tag{5}$$

where $\hbar$ is the Planck constant and $k_B$ is the Boltzmann constant.
**g.** Using equation (5) compute the entropy $S$ of a Schwarzschild black hole. Express it in terms of $c$, $G$, $\hbar$, $k_B$, and the *horizon area*, $A = 4\pi r_S^2$.
***Hint:*** Think of Einstein's famous formula.

The result from part **g.** suggests that at the classical level (i.e. ignoring quantum corrections) the total area of black holes involved in any physical process can never decrease. This is indeed true and has been formalized into a theorem by Hawking in 1971.
**h.** Using the above theorem compute the maximum amount of energy that can be radiated as gravitational waves in the merger of two Schwarzschild black holes of masses $M_1$ and $M_2$, assuming the black holes were initially at rest when far away.

We now return to the subject of Hawking temperature. While Hawking's 1974 derivation of $T_H$ was somewhat technical, in parts **i.** - **k.** we will rederive his result using a much simpler argument. Take an infinitesimal spacetime interval of the form (1) with the functions depending only on the radial coordinate $r$,

$$\left(ds\right)^2 = -c^2F(r)\left(dt\right)^2 + \frac{\left(dr\right)^2}{G(r)} + r^2\left(d\theta\right)^2 + r^2\sin^2\theta\left(d\phi\right)^2, \tag{6}$$

and suppose that $F(r)$ and $G(r)$ have a first order zero at $r_h$, that is $F(r_h) = G(r_h) = 0$, but $F'(r_h) \neq 0$ and $G'(r_h) \neq 0$. Consider only the $(t, r)$ part of $(ds)^2$ and analytically continue the coordinate time $t$ to "imaginary time" $\tau$ via $t \rightarrow i\tau$, so that the signature of the spacetime interval becomes Euclidean and the infinitesimal spacetime element $ds$ becomes an infinitesimal element of ordinary length,

$$\left(ds\right)^2 = \frac{\left(dr\right)^2}{G(r)} + c^2F(r)\left(d\tau\right)^2 \tag{7}$$

The length element $ds$ now describes how distances are measured on an ordinary 2-dimensional plane, with the origin corresponding to $r = r_h$ and $r \geq r_h$ for any point on the plane. This coordinate system is an analogue of polar coordinates, in that $r$ can be thought of as a radial coordinate, and $\tau$ as an angular coordinate that must be periodic with some period $P$.

**i.** Write down the distance $R$ from the origin to a point of coordinate $r = r_h + \varepsilon$ that is infinitesimally close to the origin. Express your answer in terms of $\varepsilon$ and $G'(r_h)$.

**j.** Write down the circumference $L$ of a circle of radial coordinate $r_h + \varepsilon$ around the origin, with $\varepsilon$ infinitesimal. Express your answer in terms of $P$, $\varepsilon$ and $F'(r_h)$.

**k.** By imposing the condition that the plane is not singular at the origin, that is that $L = 2\pi R$, determine the period $P$ of the $\tau$ coordinate. From field-theoretic arguments this period must be equal to $\hbar c/k_B T_H$. Solve for the Hawking temperature and recover equation (5) for $F(r) = G(r) = 1 - 2GM/(c^2r)$.

**l.** Compute the black hole's heat capacity $C$ in terms of $G$, $c$, $\hbar$, $k_B$ and $T_H$.

We now consider black hole evaporation. Assuming no infalling matter or energy, a black hole will slowly radiate away its mass via *Hawking radiation photons*. Although a correct treatment of the evaporation process at high energy scales requires a theory of quantum gravity, as long as $T_H$ is below the Planck scale the semi-classical approach we've been using so far suffices. In what follows we will obtain an estimate of the black hole evaporation timescale, ignoring Planck regime complications. Since at the semi-classical level the black hole spends most of its life below the Planck scale, this will be a lower bound on the estimate of the evaporation process duration.

**m.** Assuming black holes obey the blackbody law, compute the power $W$ emitted by a black hole of mass $M$. Use that the Stefan-Boltzmann constant is

$$\sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}, \tag{8}$$

and express your result in terms of $G$, $c$, $\hbar$ and $M$.

**n.** Compute the evaporation time $\tau$ in terms of $M$, assuming the result from the previous part holds at all energy scales. Compare with the age of the universe for a black hole of mass $10M_{\odot} = 2\cdot10^{31}$ kg. Use that $G = 6.67\cdot10^{-11}$ m$^3$/(s$^2$kg), $\hbar = 1.05\cdot10^{-34}$ m$^2$kg/s, $c = 3\cdot10^8$ m/s.