3^{rd} Romanian Master of Sciences 2010

Physics – Theoretical Tour

A. ELECTRICITY

a) I_F < 200 mA (the fuse is intact):

The fuse F acts as a short-circuit, and the voltage across it vanishes. No electric current I_1 flows in R_1, $I_1 = 0$.
0.25 p

No electric current flows in L_2 for $t = 0$, the voltage across the fuse is zero, the electric current I_2 is zero, $I_2 = 0$.
0.25 p

Across the inductance L_1 a constant voltage V causes I to increase at a constant rate V/L_1=1000 A/s.
0.25 p

All electrical current I flows in the fuse, $I_F=I$.
0.25 p

The melting condition is realized for $t = L_1 I_F / V$ = 0.2 ms
0.5 p

total a) 1.5 p

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b) Once the fuse melts, the current I_F vanishes, $\boldsymbol{I_F = 0}$.
0.25 p

Right after the fuse melts, the electric current I conserves its value before melting, $\boldsymbol{I}$**=200mA**.
0.25 p

The current flowing in L_2 is free of jumps (discontinuities). Then, right after melting the fuse $\boldsymbol{I_2}$**=0**.
0.25 p

As a consequence of Kirchhoff's first law, right after melting the fuse, the current I_1 flowing in R_1 is 200 mA, causing a voltage drop across R_1 of 200 V, with the "+" pole in the right hand side.
0.25 p

As a consequence, a voltage across L_1 develops $V - R_1 I_F$ =200V-10V=190V, causing the variation of I at a rate $\Delta I / \Delta t$ given by $\left(V - R_1 I_F\right) / L_1$ =**19000A/s**. Right after the fuse melts, the voltage polarity causes I to decrease at the rate above.
0.5 p

The 200 V voltage drop across R_1 produces an increase of I_2. Right after the fuse melts, the voltage drop across R_2 is zero, therefore, immediately after melting the fuse, I_2 raises at a rate of **40000A**/s.
0.25 p

From the first Kirchhoff law, it follows that immediately after the fuse melts, the current I_1 falls at a rate of **59000** A/s.
0.5 p

total b) 2.25 p

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c) For t approaching infinity:

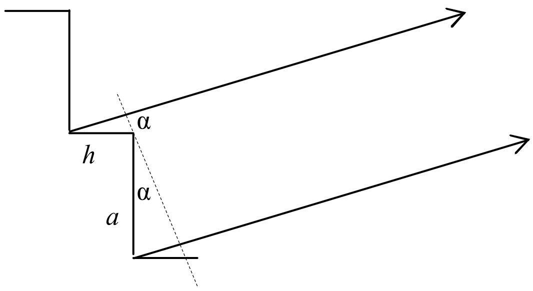
$\boldsymbol{I_1}$**=10 mA**
0.25 p

$\boldsymbol{I_2}$**=50 mA**
0.25 p

$\boldsymbol{I}$**=60 mA**
0.25 p
total c) 0.75 p

B. MICHELSON'S "LADDER"

a. 1.5p



It is obvious that for $\alpha = 0$, the path length difference δ between two neighboring light rays is $(n-1)h$. This difference increases with α.

$$\delta = nh + a\sin\alpha - h\cos\alpha = (n-1)h + a\sin\alpha + h(1-\cos\alpha) \ .$$

$$\delta = k\lambda \ , \ k \in \mathbb{N} \ .$$

One can see that in our specific example we get a diffraction maximum for $\alpha = 0$ and $k_0 = 10{,}000$. So the condition for the principal maxima can be written:

$$a\sin\alpha + h(1-\cos\alpha) = p\lambda \ , \ p = k - k_0 > 0 \ .$$

b. 1.5p

One knows that the intensity of the light diffracted by a slit with aperture a depends on the deflecting angle according to:

$$I(\alpha) = I_0 \left[\frac{\sin\left(\frac{\pi a \sin\alpha}{\lambda}\right)}{\frac{\pi a \sin\alpha}{\lambda}} \right]^2 \ .$$

The first diffraction minimum occurs for

$$\frac{\pi a \sin\alpha}{\lambda} = \pi \Rightarrow \sin\alpha = \frac{\lambda}{a} = 5 \cdot 10^{-5} \ .$$

For such a small angle,

$$\delta \approx a\alpha + h\frac{\alpha^2}{2} \approx a\alpha \approx a\sin\alpha = p\lambda \ .$$

$$\sin\alpha < \frac{\lambda}{a} \Rightarrow p < 1 \Rightarrow p = 0 \Rightarrow k = k_0 \ .$$

So the only principal maximum that can be seen is the "central" one.

c. 1.5p

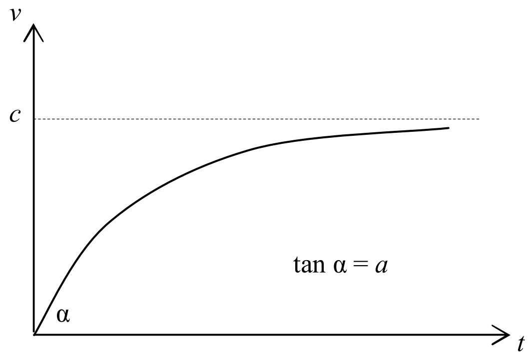
Increasing the wavelength of the light stream by $\Delta\lambda$ can lead to the overlapping of two such central maxima.

$$k_0\lambda = (k_0 - 1)(\lambda + \Delta\lambda) \Rightarrow \Delta\lambda \approx \frac{\lambda}{k_0} = 0.5\text{A} \ .$$

SPECIAL RELATIVITY: ACCELERATING SPACESHIP

All quantities in the rocket's reference frame will be denoted with a prime (').

a. 1p



b. 2p

The momentum of the rocket increases uniformly with time.

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = \text{const} \Rightarrow \frac{v_x}{\sqrt{1-\left(\frac{v_x}{c}\right)^2}} = at \Rightarrow v_x(t) = c\frac{\frac{at}{c}}{\sqrt{1+\left(\frac{at}{c}\right)^2}}.$$

(We took into account the fact that $v_x(0) = 0$.) The rocket's acceleration in the Earth's reference frame is

$$a_x(t) = \frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{a}{\left(1+\left(\frac{at}{c}\right)^2\right)^{\frac{3}{2}}}.$$

The Lorentz transformation for the accelerations on the x-axis is

$$a_x = a_x'\left(\frac{\sqrt{1-\frac{V^2}{c^2}}}{1+\frac{V}{c^2}v_x'}\right)^3.$$

For $V = v_x$, we get $v_x' = 0$, so the astronaut's "weight" will be given by

$$a_x'(t) = \frac{a_x}{\left(1-\frac{v_x^2}{c^2}\right)^{\frac{3}{2}}} = a \Rightarrow W = ma.$$

c. 1p

$$v_x = \frac{dx}{dt} \Rightarrow dx = c\frac{\frac{at}{c}}{\sqrt{1+\left(\frac{at}{c}\right)^2}}dt \Rightarrow x(t) = \frac{c^2}{a}\left[\sqrt{1+\left(\frac{at}{c}\right)^2}-1\right].$$

d. 1p

The distance increases asymptotically with time:

$$\lim_{t\to\infty}\frac{x(t)}{t}=c\ ;\ \lim_{t\to\infty}\left(x(t)-ct\right)=-\frac{c^2}{a}\ .$$

The asymptote's equation is

$$x(t)=ct-\frac{c^2}{a}=c\left(t-\frac{c}{a}\right).$$

For the drawing, see **g.**

e. 1p

As seen on the diagram, in the long run the spaceship's motion gets infinitely close to the motion of a light signal emitted from Earth at Earth time c/a.

f. 1.5p

The Earth time when the first signal reaches the rocket is given by

$$x(t)=c\left(t-\frac{c}{2a}\right)\Rightarrow t=\frac{3}{4}\frac{c}{a}\ .$$

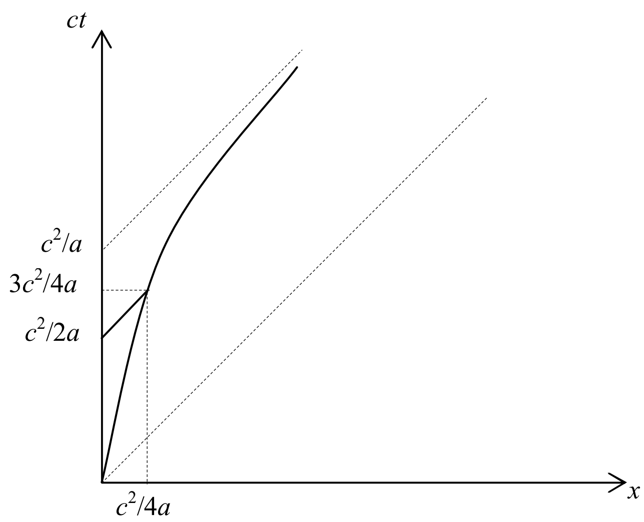
The position and the velocity of the rocket at that moment are

$$x\left(\frac{3}{4}\frac{c}{a}\right)=\frac{c^2}{4a}\ \text{ and }\ v_x\left(\frac{3}{4}\frac{c}{a}\right)=\frac{3}{5}c$$

respectively. In order to find T', for each infinitely small time interval dt in Earth's reference frame we will have to add up the corresponding time interval dt' in the rocket's reference frame.

$$\mathrm{d}t'=\mathrm{d}t\sqrt{1-\frac{v_x^2}{c^2}}=\frac{\mathrm{d}t}{\sqrt{1+\left(\frac{at}{c}\right)^2}}\Rightarrow T'=\int_0^{\frac{3}{4}\frac{c}{a}}\mathrm{d}t'=\frac{c}{a}\ln 2\approx 0.7\frac{c}{a}\ .$$

g. 0.5p



h. 1p

From the Doppler Effect formula it follows that

$$\nu' = \nu_0 \sqrt{\frac{c - v_x}{c + v_x}} = \frac{1}{2}\nu_0 \ .$$

i. 1p

Applying once again the Doppler Effect formula, we get

$$\nu = \frac{1}{4}\nu_0 \ .$$

NEWTONIAN COSMOLOGY

a. 1p

First of all, it is evident that **v** and **r** must have the same direction. Secondly, the expansion of any arbitrary two position vectors at any moment of time must preserve the already existing proportionality between them. That is,

$$\frac{r_1(t)}{r_2(t)}=\frac{r_1(t+\mathrm{d}t)}{r_2(t+\mathrm{d}t)}=\frac{r_1+v_1\mathrm{d}t}{r_2+v_2\mathrm{d}t}=\frac{v_1(t)}{v_2(t)}\ .$$

So Hubble's Law is

$$\vec{v}(t)=H(t)\cdot\vec{r}(t)\ .$$

b. 1p

Let A and B be two galaxies seen from a point in space, e.g. from Earth. According to Hubble's Law,

$$\vec{v}_{\mathrm{A}}=H(t)\cdot\vec{r}_{\mathrm{A}}\ ,$$
$$\vec{v}_{\mathrm{B}}=H(t)\cdot\vec{r}_{\mathrm{B}}\ .$$

By subtracting the two expressions, we get

$$\left(\vec{v}_{\mathrm{B}}-\vec{v}_{\mathrm{A}}\right)=H\left(\vec{r}_{\mathrm{B}}-\vec{r}_{\mathrm{A}}\right).$$

So the relative velocity of galaxy B with respect to galaxy A is proportional to its relative position with respect to that galaxy, the proportionality factor being the same Hubble constant.

c. 0.5p

Assuming that

$$\vec{r}\left(t_0\right)=\vec{v}\left(t_0\right)\cdot t_0\ ,$$

we get

$$\vec{v}\left(t_0\right)=H\left(t_0\right)\cdot\vec{v}\left(t_0\right)\cdot t_0\Rightarrow t_0=\frac{1}{H_0}\ .$$

d. 0.5p

$$\rho(t)\frac{4\pi r^3(t)}{3}=\mathrm{const}\Rightarrow\rho(t)\cdot R^3(t)\cdot r_0^3=\mathrm{const}\Rightarrow\rho(t)=\frac{\rho_0}{R^3(t)}\ .$$

e. 0.5p

$$E(t)=\frac{mv^2(t)}{2}-G\frac{m\rho(t)\frac{4\pi}{3}r^3(t)}{r(t)}=\frac{mH^2(t)r^2(t)}{2}-\frac{4\pi Gm\rho(t)r^2(t)}{3}=\frac{mR^2(t)r_0^2}{2}\left(H^2(t)-\frac{8\pi G\rho(t)}{3}\right).$$

f. 0.5p

If $\Omega>1$, the expansion will eventually come to a halt and then the universe will start to shrink until it vanishes.

If $\Omega = 1$, the universe will keep on expanding, approaching infinity with zero recessional velocity.
If $\Omega < 1$, the universe will expand to infinity with nonzero recessional velocity.

g. 0.5p

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G} \Rightarrow E(t) = \frac{mR^2(t)r_0^2H^2(t)}{2}\left(1-\Omega(t)\right).$$

Since E = const, the sign of $1 - \Omega(t)$ does not change with time.

h. 0.5p

$$v(t) = H(t)r(t) \Rightarrow H(t) = \frac{1}{r(t)}\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{R(t)}\frac{\mathrm{d}R}{\mathrm{d}t} \Rightarrow E(t) = \frac{mr_0^2}{2}\left[\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 - \frac{8\pi G\rho_0}{3R(t)}\right].$$

$$\rho_0 = \Omega_0\rho_{c0} = \frac{3\Omega_0 H_0^2}{8\pi G} \Rightarrow E(t) = \frac{mr_0^2}{2}\left[\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 - \frac{\Omega_0 H_0^2}{R(t)}\right] = \frac{mr_0^2 H_0^2\left(1-\Omega_0\right)}{2} \Rightarrow$$

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 = \frac{\Omega_0 H_0^2}{R} - H_0^2\left(\Omega_0 - 1\right) = H_0^2\left(\frac{\Omega_0}{R} + 1 - \Omega_0\right).$$

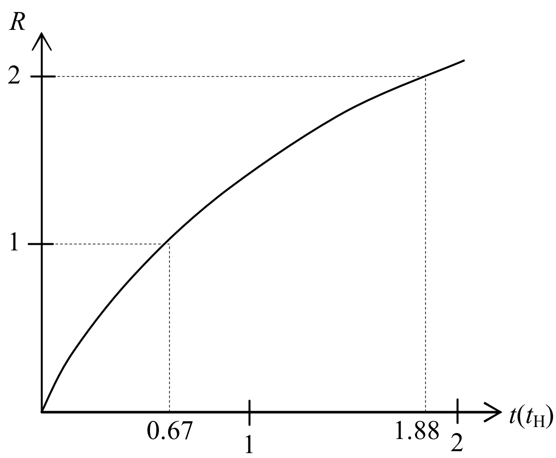
$$t \to 0 \Rightarrow R \to 0 \Rightarrow \frac{dR}{dt} \to \infty \Rightarrow RH \to \infty \Rightarrow 1 - \Omega \to 0.$$

i. 0.5p

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{H_0}{\sqrt{R}} \Rightarrow \int\sqrt{R}\,\mathrm{d}R = \int H_0\,\mathrm{d}t \Rightarrow \frac{2}{3}R^{\frac{3}{2}} = H_0 t \Rightarrow R(t) = \left(\frac{3}{2}\frac{t}{t_H}\right)^{\frac{2}{3}}.$$

Since $R_0 = 1$, we get $t_0 = 2/3\ t_H$.

j. 0.5p



k. 0.5p

$$\frac{\mathrm{d}R}{\mathrm{d}t} = H_0\sqrt{\frac{\Omega_0 - \left(\Omega_0 - 1\right)R}{R}} \Rightarrow \sqrt{\frac{R}{1 - \frac{\Omega_0 - 1}{\Omega_0}R}}\,\mathrm{d}R = H_0\sqrt{\Omega_0}\,\mathrm{d}t.$$

Now

$$x=\frac{\Omega_0-1}{\Omega_0}R\Rightarrow\left(\frac{\Omega_0}{\Omega_0-1}\right)^{\frac{3}{2}}\sqrt{\frac{x}{1-x}}dx=H_0\sqrt{\Omega_0}dt\Rightarrow\arcsin\sqrt{x}-\sqrt{x(1-x)}=\frac{H_0}{\Omega_0}(\Omega_0-1)^{\frac{3}{2}}t\ .$$

(We took into account the fact that $x(0) = 0$.) So

$$t(R)=\frac{\Omega_0}{H_0\left(\Omega_0-1\right)^{\frac{3}{2}}}\left[\arcsin\sqrt{\frac{\Omega_0-1}{\Omega_0}R}-\sqrt{\frac{\Omega_0-1}{\Omega_0}R\left(1-\frac{\Omega_0-1}{\Omega_0}R\right)}\right].$$

l. 0.5p

$$\sqrt{\frac{\Omega_0-1}{\Omega_0}R}=\sin\frac{p}{2}\Rightarrow R=\frac{\Omega_0}{\Omega_0-1}\sin^2\frac{p}{2}\Rightarrow$$

$$\begin{cases} R(p)=\dfrac{1}{2}\dfrac{\Omega_0}{\Omega_0-1}\left(1-\cos p\right); \\ t(p)=\dfrac{1}{2H_0}\dfrac{\Omega_0}{\left(\Omega_0-1\right)^{\frac{3}{2}}}\left(p-\sin p\right). \end{cases}$$

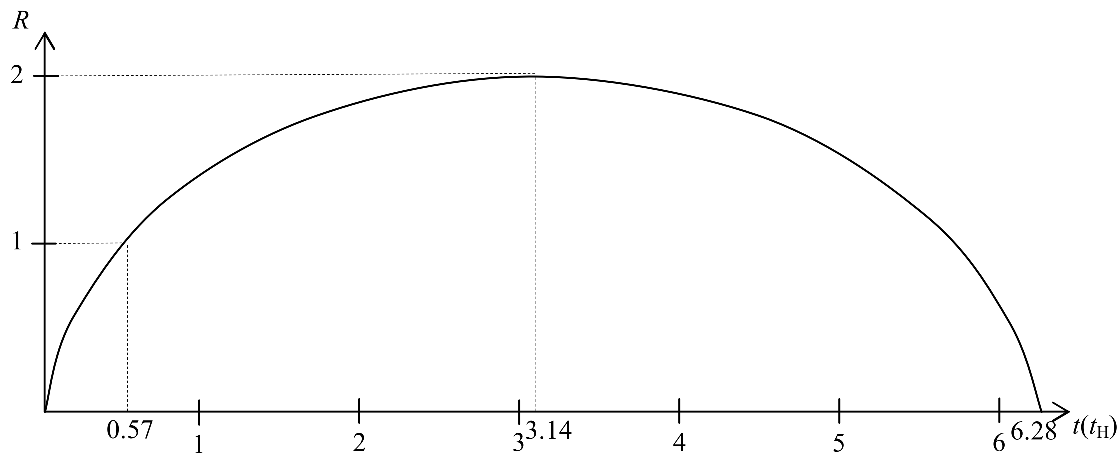
m. 0.5p

$$R(T)=0\Rightarrow 1-\cos p=0\Rightarrow p=2\pi\Rightarrow T=\pi\frac{\Omega_0}{\left(\Omega_0-1\right)^{\frac{3}{2}}}t_{\mathrm{H}}.$$

n. 0.5p

$$R_{\max}=\frac{\Omega_0}{\Omega_0-1}\ \text{for}\ p=\pi,\ \text{i.e. at}\ t=T/2\ .$$

o. 0.5p



p. 0.5p

$$\frac{\mathrm{d}R}{\mathrm{d}t}=H_0\sqrt{\frac{\Omega_0+\left(1-\Omega_0\right)R}{R}}\Rightarrow\sqrt{\frac{R}{1+\frac{1-\Omega_0}{\Omega_0}R}}\mathrm{d}R=H_0\sqrt{\Omega_0}\mathrm{d}t$$

Now

$$x=\frac{1-\Omega_0}{\Omega_0}R\Rightarrow\left(\frac{\Omega_0}{1-\Omega_0}\right)^{\frac{3}{2}}\sqrt{\frac{x}{1+x}}dx=H_0\sqrt{\Omega_0}dt\Rightarrow-\operatorname{arcsinh}\sqrt{x}+\sqrt{x(1+x)}=\frac{H_0}{\Omega_0}(1-\Omega_0)^{\frac{3}{2}}t\ .$$

(We took into account the fact that $x(0) = 0$.) So

$$t(R)=\frac{\Omega_0}{H_0(1-\Omega_0)^{\frac{3}{2}}}\left[-\operatorname{arcsinh}\sqrt{\frac{1-\Omega_0}{\Omega_0}R}+\sqrt{\frac{1-\Omega_0}{\Omega_0}R\left(1+\frac{1-\Omega_0}{\Omega_0}R\right)}\right].$$

q. 0.5p

$$\sqrt{\frac{1-\Omega_0}{\Omega_0}R}=\sinh\frac{p}{2}\Rightarrow R=\frac{\Omega_0}{1-\Omega_0}\sinh^2\frac{p}{2}\Rightarrow$$

$$\begin{cases}R(p)=\dfrac{1}{2}\dfrac{\Omega_0}{1-\Omega_0}(\cosh p-1)\ ;\\ t(p)=\dfrac{1}{2H_0}\dfrac{\Omega_0}{(1-\Omega_0)^{\frac{3}{2}}}(\sinh p-p)\ .\end{cases}$$

r. 0.25p

$$\lim_{p\to\infty}\frac{R(p)}{t(p)}=H_0\sqrt{1-\Omega_0}\Rightarrow R(t)\propto\sqrt{1-\Omega_0}\,\frac{t}{t_{\mathrm{H}}}\ .$$

s. 0.25p