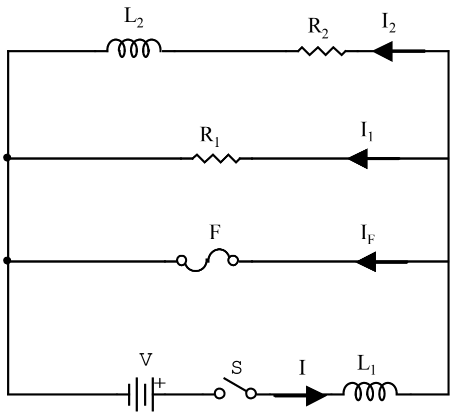
CLASSICAL PHYSICS

A. ELECTRICITY



Consider the electric circuit shown to the right. The DC battery and the two coils are ideal (they have zero electrical resistance). The fuse F has zero electrical resistance also, and melts instantaneously precisely when the current I_F reaches the value 200 mA. At the initial moment $t = 0$ all currents are zero, and the switch goes from OFF to ON.

a. Calculate and plot the currents as functions of time until the moment the fuse melts. Calculate this moment of time.
b. Calculate the values and the rates of variation for all currents in the circuit, right after the fuse melts.
c. Calculate the currents as t approaches infinity.

Use the specific values: $V = 10$ V; $L_1 = 10$ mH; $L_2 = 5$ mH; $I_F = 0.2$ A; $R_1 = 1$ kΩ; $R_2 = 200$ Ω.

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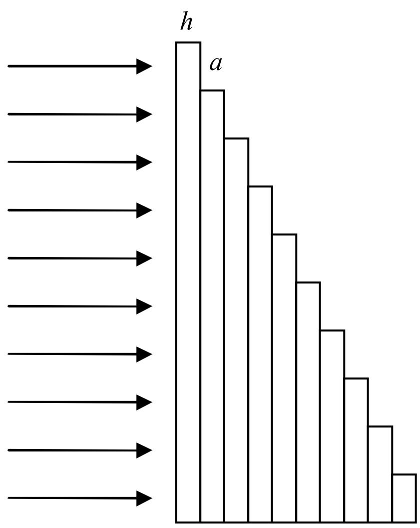
B. MICHELSON'S "LADDER"



Consider the system in the drawing acting as a diffraction grating. The system consists of a number of perfectly homogeneous and parallel glass plates, each having thickness h and refraction index n. Each plate is shorter than the previous by a.
A homogeneous monochromatic light beam with wavelength λ enters the system perpendicularly onto the longest plate. The light undergoes diffraction when leaving the system through the last contact point of each two neighboring plates.

a. Find the condition for the principal diffraction maxima in terms of a, h, n, λ and the diffraction

angle α (the deflection angle relative to the initial direction of the incident beam).
b. How many principal diffraction maxima does one get if it is assumed that these can be neatly seen only if they lay within the range of the first principal diffraction maximum of a single slit with aperture a?
c. What should the maximum spectral range $\Delta\lambda$ of the light beam be in order to avoid getting overlapping maxima of different orders?

Use the specific values: $h = a = 1\text{cm}$; $n = 1.5$; $\lambda = 500$ nm.

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SPECIAL RELATIVITY: ACCELERATING SPACESHIP

At Earth time $t = 0$, a rocket leaves Earth, starting from rest, on a space journey along a straight line. Unless otherwise stated, the rocket is to be considered point-like for all purposes. Assume the simplifying hypothesis that the thrust (force) of the engines in the Earth's frame and the rest mass of the rocket are constant in time, and neglect any gravitational and/or atmospheric influences. Let c be the speed of light in vacuum, and a the initial acceleration of the rocket.

a. Plot a qualitative graph of the rocket's speed against time as measured on Earth.
b. What is the "weight" of an astronaut with rest mass m on the spaceship?
c. Express the rocket's coordinate on the x-axis representing its trajectory as a function of Earth time t, in terms of t, c, and a. (The Earth is to be considered the origin of the axis.)
d. Draw the world line of the rocket on a space-time diagram displaying only the coordinates of interest, x and ct.
e. Determine the last moment of time t_0 at which a light signal could be emitted from Earth so that it still reaches the spaceship.

At Earth time $c/2a$, a radio station on Earth initiates a recurrent communication with the spaceship: the station emits a stream of monochromatic photons which, upon reception, are instantly reflected back towards Earth by the rocket. The photons reach the radio station, which immediately reflects them back in the direction of the spaceship, and the process repeats. When first emitted, the photons have the frequency ν_0 in the Earth's reference frame.

f. Determine the rocket time T' at which it receives for the first time a signal from Earth. (At launch, the spaceship's clocks were perfectly synchronized with those on Earth.)
g. Add to the diagram drawn in part **d** the world line of such a photon from the moment it is emitted to the moment it reaches the spaceship for the first time.
h. Find the frequency of the last stream of photons received by the spaceship.
i. Find the frequency of the last stream of photons received by the radio station.

NEWTONIAN COSMOLOGY

In this problem we will aim to describe in a very schematic way the behavior of the universe and its ultimate fate, by using simple concepts of Newtonian mechanics. In order to do this, we will have to start by assuming the validity of the so-called *Cosmological Principle*, which asserts the homogeneous and isotropic character of the universe. We will also take for granted that the universe started with the Big Bang.

The above considerations lead us to the conclusion that, seen from any point in space, the expansion of the universe must be described by a unique *scale factor*, $R(t)$, that is independent of position. That is, given an arbitrary point that currently has position vector $\mathbf{r}_0$, at some other moment in time its position was/will be characterized by

$$\vec{r}(t) = R(t) \cdot \vec{r}_0 \ .$$

(The technical term used by mathematicians for such a transformation of the space is *homothety*, but never mind that.) Obviously, at the present time t_0, $R(t_0) = 1$, and at the moment of the Big Bang, $R(0) = 0$.

The main goal of this problem is to asses the different possible kinds of evolution for $R(t)$.

1. The *Hubble Law* states the relation that has to exist between the position vector $\mathbf{r}$ and the velocity $\mathbf{v}$ of any point of the space, so that the expansion of the universe is homogeneous and isotropic. The way the velocity depends on the position vector is expressed by means of a multiplying factor having appropriate units, called the *Hubble constant*, H. It is nevertheless a function of time, just as $R(t)$ is, and we shall denote $H(t_0) = H_0$.

a. Deduce Hubble's Law.

b. Show that, in the Newtonian mechanics approximation, Hubble's Law holds for any observer considered to be "at rest" in no matter what point of the universe.

c. The current value of the Hubble constant, H_0, can be measured by experimentally determining the distance and recessional speed of a galaxy. Thus, we are in the position to make a first and crude estimate of the age of the universe, t_0. In order to do this, let us assume (incorrectly!) that the current velocities of the points of the space remained unchanged from the moment of the Big Bang until now.

Express t_0 in terms of H_0. (It is of interest to mention the fact that the observed value for H_0 leads us to an estimate for t_0 of roughly $1.38 \cdot 10^{10}$ yr. For the remaining of this problem, we will call this number the *Hubble time*, t_H.)

2. We will now study a "pressureless dust" model of the universe. By "dust" we mean that the universe contains only ordinary matter, with no radiation (photons), no neutrinos, no non-baryonic matter or anything else. By "pressureless" we understand that every point of the space is endowed with the same time-dependent density ρ, and that the total mass of the universe has a fixed value. Consequently, the density varies only as a result of the universal expansion.

d. Find the relation between $\rho(t)$ and $R(t)$ in terms of the current universal density, $\rho(t_0) = \rho_0$.

e. Consider an infinitely thin spherical layer of mass m and current radius r_0. Write down its total energy at a given moment of time in terms of m, r_0, $R(t)$, $H(t)$, $\rho(t)$, and the gravitational constant G.

f. Let us define the so-called *critical density*, ρ_c, as being the value for which the above mentioned energy is zero. The distinction between the different types of universes is made by means of a quantity called the *density parameter*, $\Omega(t)$, which represents the ratio of the actual density of the universe to the critical density at a given moment in time.

Specify how the universe behaves if $\Omega > 1$ ("closed" universe), if $\Omega = 1$ ("flat" universe), and if $\Omega < 1$ ("open" universe).

g. Express the total energy E of the spherical layer in terms of m, r_0, $R(t)$, $H(t)$, and $\Omega(t)$, and show that the character of the universe doesn't change in time.

h. Making use of the Hubble Law, find the implicit equation for the time dependence of $R(t)$ in terms of H_0 and $\Omega_0 = \Omega(t_0)$. Show that at the moment of the Big Bang the universe was essentially behaving infinitely close to a flat universe.

3. We are now in the position to make a rather more accurate estimation of the age of the universe, under the assumption of a flat universe ($\Omega_0 = 1$).

i. Solve the equation found at point **h** in order to find explicitly $R(t)$, and express t_0 in terms of t_H.

j. Draw a rough sketch of the scale factor versus time in units of t_H.

4. We will now address the more complex cases of a closed ($\Omega_0 > 1$) universe and of an open ($\Omega_0 < 1$) universe.

k. For $\Omega_0 > 1$, find the reversed dependency, of time as a function of the scale factor, in terms of H_0 and Ω_0.

Hint:

$$\int \sqrt{\frac{x}{1-x}}dx = \arcsin(\sqrt{x}) - \sqrt{x(1-x)} + C\,.$$

l. In order to be able to study $R(t)$, we will try to express this function in a parametric way. To do this, let us denote the *arcsin* function in the above solution by *p/2*.

Write down $R(p)$ and $t(p)$.

m. Express the age of the universe T at the final moment of the Big Crunch (the opposite of the Big Bang) in terms of t_H and Ω_0.

n. Find the maximum size (i.e. maximum R) of the universe in terms of Ω_0.

o. For $\Omega_0 = 2$, draw a rough sketch of R versus t in units of t_H.

p. For $\Omega_0 < 1$, find the reversed dependency $t(R)$, in terms of H_0 and Ω_0.

Hint:

$$\int \sqrt{\frac{x}{1+x}}dx = -\text{arcsinh}(\sqrt{x}) + \sqrt{x(1+x)} + C\,.$$

q. Denote the above *arcsinh* function by *p/2*, and write down $R(p)$ and $t(p)$.

r. Show that, in the long run, the expansion of the universe will stabilize itself infinitely close to a uniform rate, and find that rate in terms of H_0 and Ω_0.

s. For $\Omega_0 = 0.5$, draw a rough sketch of $R(t)$ in units of t_H.